# QUANTUM FIELD THEORIES ON ALGEBRAIC CURVES AND A. WEIL RECIPROCITY LAW

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Abstract. Using Serre's adelic interpretation of the cohomology, we develop "differential and integral calculus" on an algebraic curve $X$ over an algebraically closed constant field $k$ of characteristic zero, define an algebraic analogs of additive and multiplicative multi-valued functions on $X$, and prove corresponding generalized residue theorem and A. Weil reciprocity law. Using the representation theory of global Heisenberg and lattice Lie algebras and the Heisenberg system, we formulate quantum field theories of additive, charged, and multiplicative bosons on an algebraic curve $X$. We prove that extension of the respected global symmetries — Witten's additive and multiplicative Ward identities — from the $k$-vector space of rational functions on $X$ to the $k$-vector space of additive multi-valued functions, and from the multiplicative group of rational functions on $X$ to the group of multiplicative multi-valued functions on $X$, defines these theories uniquely. The quantum field theory of additive bosons is naturally associated with the algebraic de Rham theorem and the generalized residue theorem, and the quantum field theory of multiplicative bosons — with the generalized A. Weil reciprocity law.

## 1. Introduction

Classical theory of compact Riemann surfaces has an algebraic counterpart, theory of of algebraic functions in one variable over an arbitrary constant field, developed by R. Dedekind and H. Weber. Introduction of differentials into the algebraic theory by E. Artin and H. Hasse, and of ideles and adeles by C. Chevalley and A. Weil, open the way for application of infinite-dimensional methods to the theory of algebraic curves. Classic examples of using such methods are J.-P. Serre adelic interpretation of cohomology [Ser88], and J. Tate proof of the general residue theorem [Tat68]. In 1987, E. Arbarello, C. de Concini and V. Kac [ADCK89] interpreted Tate's approach in terms of central extensions of infinite-dimensional Lie algebras, and proved the celebrated A. Weil reciprocity law on algebraic curves by using the infinite-wedge representation.

In 1987 D. Kazhdan [Kaz87] and E. Witten [Wit88a] proposed an adelic formulation of the quantum field theory of one-component free fermions on an algebraic curve, and in [Wit88b] E. Witten outlined the approach toward other quantum field theories. Let $X$ be an algebraic curve over an algebraically closed constant field $k$, and let $L$ be a spin structure on $X$.

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Denote by $\mathcal{M}(L)$ the infinite-dimensional $k$-vector space of meromorphic sections of $L$ over $X$, and by $\mathcal{M}_P$ — the completions of $\mathcal{M}(L)$ at all points $P \in X$. The approach in [Kaz87, Wit88a] can be succinctly summarized as follows.

- The global Clifford algebra $\mathrm{Cl}_X$ on $X$, a restricted direct product over all points $P \in X$ of local Clifford algebras $\mathrm{Cl}_P$, associated with the $k$-vector spaces $\mathcal{M}_P$ by the residue maps $\mathrm{Res}_P(fg)$.
- The adelic Clifford module — global fermion Fock space $\mathfrak{F}_X$ — a restricted $\mathbb{Z}/2\mathbb{Z}$-graded symmetric tensor product of the local Clifford modules $\mathfrak{F}_P$ over all $P \in X$.
- The "expectation value" functional, the linear map $\langle \cdot \rangle : \mathfrak{F}_X \to k$, satisfying

$$\langle f \cdot u \rangle = 0 \quad \text{for all} \quad f \in \mathcal{M}(L) \subset \mathrm{Cl}_X,\ u \in \mathfrak{F}_X, \tag{1.1}$$

  where the vector space $\mathcal{M}(L)$ is embedded diagonally into the global Clifford algebra $\mathrm{Cl}_X$.

In this pure algebraic formulation of one-component free fermions on an algebraic curve, "products of field operators inserted at points $P \in X$" are replaced by vectors $u = \hat{\otimes}_{P \in X} u_P \in \mathfrak{F}_X$, and the linear map $\langle \cdot \rangle$ is a mathematical way of defining "correlation functions of quantum fields", which at the physical level of rigor are introduced by Feynman path integral. The vector space $\mathcal{M}(L)$ acts on $\mathfrak{F}_X$ by "global symmetries", and invariance of the quantum theory of free fermions with respect to this symmetry is expressed by "quantum conservation laws" (1.1), also called the additive Ward identities. It is proved in [Wit88a, Wit88b] that if the spin structure $L$ has no global holomorphic sections, then the additive Ward identities determine the expectation value functional $\langle \cdot \rangle$ uniquely. Moreover, (1.1) is compatible with the global residue theorem

$$\sum_{P \in X} \mathrm{Res}_P(f dg) = 0, \quad f, g \in \mathcal{M}(L).$$

In [Wit88b], E. Witten developed the rudiments of quantum field theories associated with a "current algebra on an algebraic curve", and mentioned the theories associated with a "loop group on an algebraic curve". Corresponding global symmetries of these theories are, respectively, rational maps of an algebraic curve $X$ into a finite-dimensional semi-simple Lie algebra over the field $k$, and rational maps of $X$ into the corresponding Lie group. In the latter case, the analog of quantum conservation laws (1.1) was called "multiplicative Ward identities" in [Wit88b]. However, as it was emphasized in [Wit88b, Sect. IV], when the genus of $X$ is greater than zero, the Ward identities, even in the Lie-algebraic case, do not determine the expectation value functional $\langle \cdot \rangle$ uniquely. Thus the main problem of constructing quantum field theories on an algebraic curve is to find additional conditions which would determine the linear functional $\langle \cdot \rangle$ uniquely.

In [Tak00] we announce solution of this problem for the theories in the simplest scalar case when the finite-dimensional Lie algebra is the abelian Lie algebra $k$, and the corresponding Lie group is the multiplicative group $k^* = k \setminus \{0\}$. We call these theories, respectively, quantum field theories of additive and multiplicative bosons. The proposed solution in [Tak00] was to enlarge the global symmetries by considering algebraic analogs of the vector space "multi-valued additive functions on a Riemann surface" — the analogs of classical abelian integrals of the second kind with zero $a$-periods — and the group of "multi-valued multiplicative functions on a Riemann surface" — the analogs of exponentials of abelian integrals of the third kind with zero $a$-periods. Though classical theory of abelian integrals on compact Riemann surface has been already developed by Riemann (see, e.g., [Iwa93] and [Kra72] for the modern exposition), corresponding algebraic theory — "integral calculus on algebraic curves" — has not been fully developed. In the present paper we fill this gap for the case when the constant field $k$ has characteristic zero, and give an explicit construction of quantum field theories of additive and multiplicative bosons on an algebraic curve. These theories are naturally associated with the algebraic de Rham theorem and A. Weil reciprocity law, and their corresponding global symmetries are, respectively, the vector space of additive multi-valued functions and the group of multiplicative multi-valued functions.

Here is the more detailed content of the paper. For the convenience of the reader, in Section 2 we recall necessary facts of the theory of algebraic curves. Namely, let $X$ be an algebraic curve of genus $g$ over an algebraically closed constant field $k$, $F = k(X)$ be the field of rational functions on $X$, and $F_P$ be the corresponding local fields — completions of the field $F$ with respect to the regular discrete valuations $v_P$ corresponding to the discrete valuation rings at points $P \in X$. In Section 2.1 we introduce the ring of adeles

$$\mathbb{A}_X = \coprod_{P\in X} F_P$$

— a restricted direct product of the local fields $F_P$ — and present Serre's adelic interpretation of the cohomology. In Section 2.2 we recall the definitions of the $F$-module $\Omega^1_{F/k}$ of Kähler differentials on $X$, of the corresponding $\mathbb{A}_X$-module of differential adeles $\boldsymbol{\Omega}_X$, and of the differential and residue maps $d : \mathbb{A}_X \to \boldsymbol{\Omega}_X$ and $\operatorname{Res} : \boldsymbol{\Omega}_X \to k$. In Section 2.3 we present the Serre duality and the Riemann-Roch theorem, and in Section 2.4 we define the group of ideles $\mathbb{J}_X$, the local and global tame symbols, and state A. Weil reciprocity law.

In Section 3, assuming that the constant field $k$ has characteristic zero, we recall the "differential calculus" on an algebraic curve $X$ — the structure theory of the $k$-vector space of Kähler differentials $\Omega^1_{F/k}$ on $X$ — and develop the corresponding "integral calculus". Namely, in Section 3.1, following Chevalley [Che63] and Eicher [Eic66], for the $k$-vector space $\Omega^{(2\mathrm{nd})}$ of

the differentials of the second kind — differentials on $X$ with zero residues — we introduce the skew-symmetric bilinear form $(\omega_1, \omega_2)_X$ by

$$(\omega_1, \omega_2)_X = \sum_{P \in X} \operatorname{Res}_P(d^{-1}\omega_1\,\omega_2), \quad \omega_1, \omega_2 \in \Omega^{(2\mathrm{nd})}.$$

The main result of the differential calculus is Theorem 3.1, the algebraic version of the de Rham theorem. It goes back to Chevalley and Eichler, and for the algebraic curve $X$ of genus $g \geq 1$[1] states that $2g$-dimensional $k$-vector space $\Omega^{(2\mathrm{nd})}/dF$ is a symplectic vector space with the symplectic form $(\ ,\ )_X$. Moreover, for every choice of a degree $g$ non-special effective divisor $D = P_1 + \cdots + P_g$ on $X$, together with the uniformizers $t_i$ at the points $P_i$, there is an isomorphism

$$\Omega^{(2\mathrm{nd})}/dF \simeq \Omega^{(2\mathrm{nd})} \cap \Omega^1_{F/k}(-2D)$$

and a symplectic basis $\{\theta_i, \omega_i\}_{i=1}^g$ of $\Omega^{(2\mathrm{nd})} \cap \Omega^1_{F/k}(-2D)$, consisting, respectively, of differentials of the first and second kinds $\theta_i$ and $\omega_i$ with the following properties. Differentials $\theta_i$ vanish at all points $P_j$ for $j \neq i$, and $\theta_i = (1 + O(t_i))dt_i$ at $P_i$, whereas differentials $\omega_i$ are regular at all points $P_j$ for $j \neq i$, and $\omega_i = (t_i^{-2} + O(t_i))dt_i$ at $P_i$. The differentials $\theta_i$ and $\omega_i$ are, respectively, algebraic analogs of differentials of the first kind on a compact Riemann surface with normalized "$a$-periods", and differentials of the second kind with second order poles, "zero $a$-periods" and normalized "$b$-periods". Algebraically, the $a$-periods of $\omega \in \Omega^{(2\mathrm{nd})}$ are defined by $(\omega, \omega_i)_X$, $i = 1, \dots, g$, and we denote by $\Omega_0^{(2\mathrm{nd})}$ the isotropic subspace of $\Omega^{(2\mathrm{nd})}$ consisting of differentials of the second kind with zero $a$-periods. According to Proposition 3.1,

$$\Omega_0^{(2\mathrm{nd})} = k \cdot \omega_1 \oplus \cdots \oplus k \cdot \omega_g \oplus dF. \tag{1.2}$$

In Section 3.1 we also introduce the algebraic notion of additive multi-valued functions on $X$. By definition, the $k$-vector space of additive multi-valued functions is a subspace $\mathcal{A}(X)$ of the adele ring $\mathbb{A}_X$ satisfying $F \subset \mathcal{A}(X)$ and $d\mathcal{A}(X) \subset \Omega^1_{F/k}$, and the additional property that if $a \in \mathcal{A}(X)$ is such that $da = df$ for some $f \in F$, then $a - f = c \in k$. The main result of the integral calculus for differentials of the second kind with zero $a$-periods is explicit construction of the vector space $\mathcal{A}(X, D)$ in Example 3.1, which plays a fundamental role for the theory of additive bosons. It is parametrized by the choice of the degree $g$ non-special divisor $D = P_1 + \cdots + P_g$ on $X$, the uniformizers $t_i$ at the points $P_i$, and the solutions in $\mathbb{A}_X$ of the equations $d\eta_i = \omega_i$ (with any fixed choice of local additive constants). It is defined by

$$\mathcal{A}(X; D) = k \cdot \eta_1 \oplus \cdots \oplus k \cdot \eta_g \oplus F \subset \mathbb{A}_X$$

and satisfies the property $d(\mathcal{A}(X; D)) = \Omega_0^{(2\mathrm{nd})}$. Finally, we introduce additive multi-valued functions $\eta_P^{(n)} \in \mathcal{A}(X; D)$ with single poles at $P \in X$

[1] The case $g = 0$ is trivial.

of any given order $n$, and in Lemma 3.1 prove that every $f \in F$ admits a unique partial fraction expansion with simple fractions given by these $\eta_P^{(n)}$.

Section 3.2 is devoted to the differential and integral calculus for the differentials of the third kind on $X$ — Kähler differentials with only simple poles. We define the $a$-periods of a third kind differential $\omega$ by

$$-(\omega_i, \omega)_X = -\sum_{P\in X} \mathrm{Res}_P(\eta_i\omega), \quad i = 1, \dots g,$$

and prove that there is a choice of local additive constants in the definition of $\eta_i = d^{-1}\omega_i$ such that all logarithmic differentials $d\log f = \dfrac{df}{f}$, $f \in F^*$, have zero $a$-periods; all such choices are parametrized by the $g$ elements in $\mathrm{Hom}(\mathrm{Pic}_0(X), k)$. For every $P, Q \in X$, $P \neq Q$, denote by $\omega_{PQ}$ the unique differential of the third kind with simple poles at points $P$ and $Q$ with respective residues 1 and $-1$ and zero $a$-periods. The differentials $\omega_{PQ}$ span the vector space $\Omega_0^{\mathrm{3rd}}$ of the differentials of the third kind with zero $a$-periods, and

$$d\log f = \sum_{i=1}^{n} \omega_{P_iQ_i}, \tag{1.3}$$

where $(f) = \sum_{i=1}^n (P_i - Q_i)$ is the divisor of $f \in F^*$. The main result of the integral calculus for the differentials of the third kind, summarized in Proposition 3.2, is the existence of an algebraic analog of the classical Schottky-Klein prime form on a compact Riemann surface — the family of ideles $e_Q = \{e_{Q,P}\}_{P\in X} \in \mathbb{J}_X$ parametrized by points $Q \in X$ with the following properties.

- For all $P \in X$, the elements $e_{Q,P} \in F_P^*$ satisfy $v_P(e_{Q,P}) = 0$, when $P \neq Q$ , and $v_Q(e_{Q,Q}) = 1$ when $P = Q$.
- For all $P, Q \in X$, $P \neq Q$, the constants $c_{Q,P} = e_{Q,P} \bmod \mathfrak{p} \in k^*$ satisfy $c_{Q,P} = -c_{P,Q}$.
- For all $P, Q \in X$, $P \neq Q$,
$$\omega_{PQ} = d\log f_{PQ}, \quad \text{where} \quad f_{PQ} = \frac{e_P}{e_Q} \in \mathbb{J}_X.$$
- For every $f \in F^*$ with $(f) = \sum_{i=1}^n (P_i - Q_i)$,
$$f = c\prod_{i=1}^{n} f_{P_iQ_i} \quad \text{where} \quad c = c(f) \in k^*.$$

The latter property is a unique factorization of rational functions on $X$ into the products of "elementary functions" $f_{PQ}$, which should be considered as an integral form of the differential property (1.3). Correspondingly, the algebraic analogs of the exponentials of abelian integrals of the third kind with zero $a$-periods are defined by

$$\exp\int_P^Q \omega_{RS} = \frac{f_{RS}(Q)}{f_{RS}(P)}.$$

They satisfy classical "exchange law of variable and parameter", proved in Lemma 3.3.

In Section 3.2 we also introduce the algebraic notion of multiplicative multi-valued functions on $X$. By definition, a group of multiplicative multi-valued functions on $X$ is a subgroup $\mathcal{M}(X)$ of the idele group $\mathbb{J}_X$ satisfying $F^* \subset \mathcal{M}(X)$ and $d\log m = \dfrac{dm}{m} = \omega \in \Omega^1_{F/k}$ for every $m \in \mathcal{M}(X)$, and the additional property that if $m \in \mathcal{M}(X)$ is such that $d\log m = d\log f$ for some $f \in F^*$, then $m = cf$, $c \in k^*$. The main result of the integral calculus for differentials of the third kind with zero $a$-periods is a construction of the subgroup $\mathcal{M}(X, D)$ in Example 3.2, which plays a fundamental role in the theory of multiplicative bosons. Namely, it is a subgroup of $\mathbb{J}_X$ generated by the ideles $f_{PQ}$ for all $P, Q \in X$, $P \neq Q$, and it is associated with the vector space $\mathcal{A}(X, D)$, defined in Example 3.1. Finally, in Proposition 3.3 we show that the restriction of the global tame symbol to the subgroup $\mathcal{M}(X, D) \times \mathcal{M}(X, D)$ of $\mathbb{J}_X \times \mathbb{J}_X$ is the identity, which can be considered as generalized A. Weil reciprocity law for multiplicative functions.

In Section 4 we formulate local quantum field theories of additive, charged and multiplicative bosons. The local theory of additive bosons is associated with the representation theory of the local Heisenberg algebra $\mathfrak{g}_P$ — a one-dimensional central extension of the abelian Lie algebra $F_P$, $P \in X$, by the 2-cocycle $c_P(f, g) = -\operatorname{Res}_P(f dg)$. In Section 4.1 we introduce the highest weight representation $\rho$ of $\mathfrak{g}_P$ in the local Fock space $\mathscr{F}_P$, and define the corresponding contragradient representation $\rho^\vee$ of $\mathfrak{g}_P$ in the dual local Fock space $\mathscr{F}_P^\vee$. In Section 4.2 we define a local lattice algebra $\mathfrak{l}_P$ — a semi-direct sum of the local Heisenberg algebra $\mathfrak{g}_P$ and the abelian Lie algebra $k[\mathbb{Z}]$, the group algebra of $\mathbb{Z}$. Corresponding irreducible highest weight $\mathfrak{l}_P$-module is the local Fock space $\mathscr{B}_P$ of "charged bosons" — a symmetric tensor product of $k[\mathbb{Z}]$ and $\mathscr{F}_P$. Material in Sections 4.1 and 4.2 is essentially standard and can be found in [Kac90, FBZ04]. Finally, in Section 4.3 we, following H. Garland and G. Zuckerman [GZ91], introduce local quantum field theory of multiplicative bosons. It is given by the so-called Heisenberg system — a triple $(G_P, \mathfrak{g}_P, \mathrm{Ad})$ — where $G_P$ is the central extension of the multiplicative group $F_P^*$ by the local tame symbol, and Ad stands for a certain "adjoint action" of $G_P$ on $\mathfrak{g}_P$ (Ad is well-defined despite the fact that the Heisenberg algebra $\mathfrak{g}_P$ is not a Lie algebra of $G_P$). A representation of the Heisenberg system $(G_P, \mathfrak{g}_P, \mathrm{Ad})$ in a $k$-vector space $V$ is a pair $(R_P, dR_P)$, where $R_P$ is a representation of the group $G_P$, and $dR_P$ is a representation of the Lie algebra $\mathfrak{g}_P$, satisfying $dR_P(\mathrm{Ad}\, g \cdot x) = R_P(g) dR_P(x) R_P(g)^{-1}$. Following [GZ91], we define a representation of the local Heisenberg system $(G_P, \mathfrak{g}_P, \mathrm{Ad})$ in the local Fock space $\mathscr{B}_P$ of charged bosons, as well as the corresponding contragradient representation in the dual Fock space $\mathscr{B}_P^\vee$.

Finally, in Section 5 we formulate global quantum field theories, starting in Section 5.1 with the theory of additive bosons on an algebraic curve $X$. The latter is associated with a "current algebra on an algebraic curve" —

the global Heisenberg algebra $\mathfrak{g}_X$ — the one-dimensional central extension of the abelian Lie algebra $\mathfrak{gl}_1(\mathbb{A}_X) = \mathbb{A}_X$ by the 2-cocycle $c_X = \sum_{P\in X} c_P$. Since the $\Omega_0^{\text{2nd}}$ is the isotropic subspace with respect to the bilinear form $(\ ,\ )_X$, we have

$$c_X(a_1, a_2) = 0 \text{ for all } a_1, a_2 \in \mathcal{A}(X, D) \subset \mathbb{A}_X, \tag{1.4}$$

which can be considered as generalized residue theorem for the additive multi-valued functions. The irreducible highest weight module of the global Heisenberg algebra $\mathfrak{g}_X$ is the global Fock space $\mathscr{F}_X$, a restricted symmetric tensor product of local Fock spaces $\mathscr{F}_P$ over all points $P \in X$. The global Fock space can be considered as "the space of observables of the quantum field theory of additive bosons" on an algebraic curve. In Theorem 5.1 we prove that there exists a unique normalized expectation value functional $\langle\cdot\rangle : \mathscr{F}_X \to k$, uniquely characterized by the global symmetries

$$\langle \rho(a) v\rangle = 0 \quad \text{for all} \quad a \in \mathcal{A}(X; D) \text{ and } v \in \mathscr{F}_X, \tag{1.5}$$

where the subspace $\mathcal{A}(X; D) \subset \mathbb{A}_X$ is a vector space of additive multi-valued functions on $X$, defined in Section 3.1, and $\rho : \mathfrak{g}_X \to \operatorname{End}\mathscr{F}_X$ is the corresponding representation of the global Heisenberg algebra. Specifically, we show that

$$\langle v\rangle = (\Omega_X, v) \text{ for all } v \in \mathscr{F}_X,$$

where $\Omega_X \in \mathscr{F}_X^\vee$ — the dual Fock space to $\mathscr{F}_X$ — satisfies an infinite system of equations

$$\Omega_X \cdot \rho^\vee(a) = 0 \text{ for all } a \in \mathcal{A}(X, D). \tag{1.6}$$

The vector $\Omega_X$ is given by an explicit formula (see Theorem 5.1), which encodes all "correlation functions of the quantum field theory of additive bosons" on an algebraic curve $X$. The reciprocity law for the differentials of the second kind with zero $a$-periods, proved in Lemma 3.1, plays a fundamental role of ensuring compatibility of the system (1.6). The additive Ward identities (1.5) are also compatible with the generalized residue theorem. Namely, since $[\rho(x), \rho(y)] = c_X(x, y)\boldsymbol{I}$ for $x, y \in \mathbb{A}_X$, where $\boldsymbol{I}$ is the identity operator in $\mathscr{F}_X$, we get from (1.5) that for $a_1, a_2 \in \mathcal{A}(X, D)$,

$$0 = \langle(\rho(a_1)\rho(a_2) - \rho(a_2)\rho(a_1))v\rangle = c_X(a_1, a_2)\langle v\rangle \text{ for all } v \in \mathscr{F}_X,$$

which gives (1.4).

In Section 5.2 we define a global lattice algebra $\mathfrak{l}_X$ as a semi-direct sum of the global Heisenberg algebra $\mathfrak{g}_X$ and the abelian Lie algebra $k[\operatorname{Div}_0(X)]$ with generators $e_D$, $D \in \operatorname{Div}_0(X)$ — the group algebra of the additive group $\operatorname{Div}_0(X)$ of degree 0 divisors on $X$. Its irreducible highest weight module is the global Fock space $\mathscr{B}_X$ of charged bosons — the symmetric tensor product of the group algebra $k[\operatorname{Div}_0(X)]$ and the Fock space of additive bosons $\mathscr{F}_X$. The main result of this section is Theorem 5.2. It states that there is a unique expectation value functional $\langle\cdot\rangle : \mathscr{B}_X \to k$, which is normalized with respect to the action of the group algebra $k[\operatorname{Div}_0(X)]$ and satisfies additive Ward identities (1.5) with respect to the action of global symmetries —

additive multi-valued functions $\mathcal{A}(X,D)$ — in the global Fock space $\mathscr{B}_X$. It has the form $\langle v\rangle = (\hat{\Omega}_X, v)$ where $\hat{\Omega}_X \in \mathscr{B}_X^\vee$ — the dual Fock space to $\mathscr{B}_X$ — is given by an explicit formula (see Theorem 5.2), which encodes all "correlation functions of the quantum field theory of charged additive bosons" on an algebraic curve $X$.

In the last Section 5.3 we formulate quantum field theory of multiplicative bosons. We define the global Heisenberg system $(G_X, \mathfrak{g}_X, \mathrm{Ad})$, where $G_X$ is a central extension of the group of ideles $\mathbb{J}_X$ by the global tame symbol $\tau_X$, and in Theorem 5.3 we construct its representation $(R_X, dR_X)$ in the global Fock space $\tilde{\mathscr{B}}_X$— the symmetric tensor product of the group algebra $k[\mathrm{Div}(X)]$ and $\mathscr{F}_X$. For the subgroup $G_X^0$ — the central extension of the subgroup $\mathbb{J}_X^0$ of degree 0 ideles — the representation $R_X$ has an invariant subspace $\mathscr{B}_X$. In Theorem 5.3 we prove that there is a unique normalized expectation value functional $\langle\,\cdot\,\rangle : \mathscr{B}_X \to k$, satisfying global symmetries

$$\langle dR_X(a)v\rangle = 0 \quad\text{and}\quad \langle R_X(m)v\rangle = \langle v\rangle \tag{1.7}$$

for all $a \in \mathcal{A}(X;D) \subset \mathbb{A}_X$, $m \in \mathcal{M}(X;D) \subset \mathbb{J}_X^0$, and $v \in \mathscr{B}_X$. The latter relations in (1.7) are the multiplicative Ward identities in the sense of E. Witten [Wit88b] (see also [SR88]). As in previous sections, the expectation value functional has the form $\langle v\rangle = (\boldsymbol{\Omega}_X, v)$, where $\boldsymbol{\Omega}_X \in \mathscr{B}_X^\vee$ is given by an explicit formula (see Theorem 5.3), which encodes all "correlation functions of the quantum field theory of multiplicative bosons" on an algebraic curve $X$. The property $c_{P,Q} = -c_{Q,P}$ of the algebraic analog of the prime form, which is fundamental for the exchange law of variable and parameter, proved in Lemma 3.3, ensures the compatibility of the infinite system of equations for determining $\boldsymbol{\Omega}_X$. The multiplicative Ward identities are also compatible with the generalized A. Weil reciprocity law for multiplicative multi-valued functions, proved in Proposition 3.3. Namely, since $R_X(ab) = \tau_X(a,b)R_X(a)R_X(b)$ for $a, b \in \mathbb{J}_X$, we get from (1.7) that for all $m_1, m_2 \in \mathcal{M}(X,D)$ and $v \in \mathscr{B}_X$,

$$\langle v\rangle = \langle R_X(m_1m_2)v\rangle = \tau_X(m_1,m_2)\langle R_X(m_1)R_X(m_2)v\rangle = \tau_X(m_1,m_2)\langle v\rangle,$$

so that $\tau_X(m_1,m_2) = 1$.

Finally, we note that our construction of quantum field theories on algebraic curves can be considered as an algebraic counterpart of geometric realization of conformal field theories on Riemann surfaces in [KNTY88]. In particular, explicit formula for the vector $\boldsymbol{\Omega}_X \in \mathscr{B}_X^\vee$ in quantum field theory of multiplicative bosons contains all correlation functions of vertex operators in [KNTY88, Sect. 6A].

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## 2. Basic Facts

Here we recall necessary facts from the theory of algebraic curves. The material is essentially standard and can be found in [Che63, Ser88, Iwa93].

2.1. **Definitions.** An algebraic curve $X$ over an algebraically closed field $k$ is an irreducible, nonsingular, projective variety over $k$ of dimension 1, equipped with Zariski topology. The field $F = k(X)$ of rational functions on $X$ is a finitely-generated extension of the field $k$ of the transcendence degree 1. Conversely, every finitely-generated extension of $k$ of the transcendence degree 1 corresponds to the unique, up to an isomorphism, algebraic curve over $k$. Closed points $P$ on $X$ correspond to discrete valuation rings $O_P$, the subrings of the field $F$. The rings $O_P$ for all points $P \in X$ form a sheaf of rings over $X$ — the structure sheaf $\mathcal{O}_X$, a subsheaf of the constant sheaf $\underline{F}$.

For every point $P \in X$ let $v_P$ be the regular discrete valuation of the field $F$ over $k$, corresponding to the discrete valuation ring $O_P$. Completion of the field $F$ with respect to $v_P$ is the complete closed field $F_P$ with the valuation ring $\mathcal{O}_P$ — the completed local ring at $P$, the prime ideal $\mathfrak{p}$, and the residue class field $k = \mathcal{O}_P/\mathfrak{p}$. The ring of adeles $\mathbb{A}_X$ of an algebraic curve $X$,

$$\mathbb{A}_X = \coprod_{P \in X} F_P,$$

is a restricted direct product over all points $P \in X$ of the local fields $F_P$ with respect to the local rings $\mathcal{O}_P$. By definition,

$$x = \{x_P\}_{P \in X} \in \mathbb{A}_X \text{ if } x_P \in \mathcal{O}_P \text{ for all but finitely many } P \in X.$$

The field $F$ is contained in all local fields $F_P$ and is diagonally embedded into $\mathbb{A}_X$ by

$$F \ni f \mapsto \{f|_P\}_{P \in X} \in \mathbb{A}_X.$$

The divisor group $\mathrm{Div}(X)$ of $X$ is a free abelian group generated by points $P \in X$. By definition,

$$D = \sum_{P \in X} n_P \cdot P \in \mathrm{Div}(X)$$

if $n_P = v_P(D) \in \mathbb{Z}$, and $n_P = 0$ for all but finitely many $P \in X$. The divisors of the form

$$(f) = \sum_{P \in X} v_P(f) \cdot P \in \mathrm{Div}(X),$$

where $f \in F^* = F \setminus \{0\}$, the multiplicative group of the field $F$, are called principal divisors. The principal divisors form a subgroup $\mathrm{PDiv}(X) \simeq F^*/k^*$ of the divisor group $\mathrm{Div}(X)$. The degree of a divisor $D$ is

$$\deg D = \sum_{P \in X} n_P = \sum_{P \in X} v_P(D) \in \mathbb{Z},$$

and $\deg(f) = 0$ for $f \in F^*$. A divisor $D$ is said to be effective, if $v_P(D) \geq 0$ for all $P \in X$. By definition, divisors $D_1$ and $D_2$ are linear equivalent,

$D_1 \sim D_2$, if $D_1 - D_2 = (f)$, $f \in F^*$. The equivalence classes of divisors form the divisor class group $\mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{PDiv}(X)$.

For every divisor $D$ the subspace $\mathbb{A}_X(D)$ of the $k$-vector space $\mathbb{A}_X$ is defined by

$$\mathbb{A}_X(D) = \{x \in \mathbb{A}_X : v_P(x_P) \geq -v_P(D) \text{ for all } P \in X\}.$$

The ring of adeles $\mathbb{A}_X$ is a topological ring with the product topology. The base of neighborhoods of 0 is given by the subspaces $\mathbb{A}_X(D)$, $D \in \mathrm{Div}(X)$, and $\mathbb{A}_X$ is a $k$-vector space with linear topology in the sense of Lefschetz [Lef42, Ch. II, §6]. Every subspace $\mathbb{A}_X(D)$ is linear compact, so that $\mathbb{A}_X$ is locally linear compact. The $k$-vector space $F = k(X)$ is discrete in $\mathbb{A}_X$ and the quotient space $\mathbb{A}_X/F$ is linear compact [Iwa93, App., §3].

To every divisor $D$ there corresponds an algebraic coherent sheaf $\mathcal{F}(D)$ on $X$ — a subsheaf of the constant sheaf $\underline{F}$ whose stalk at each point $P \in X$ is

$$\mathcal{F}(D)_P = \{f \in F : v_P(f) \geq -v_P(D)\}.$$

Linear equivalent divisors correspond to the isomorphic sheaves. Denote by $H^i(X, \mathcal{F}(D))$ the Čech cohomology groups of the sheaf $\mathcal{F}(D)$ — a finite-dimensional vector spaces over $k$ that vanish for $i > 1$ — and put $h^i(D) = \dim_k H^i(X, \mathcal{F}(D))$. The zero divisor $D = 0$ corresponds to the structure sheaf $\mathcal{O}_X$. In this case, $h^0(0) = 1$ and $h^1(0) = g$ — the arithmetic genus of the algebraic curve $X$. One has

$$H^0(X, \mathcal{F}(D)) = \mathbb{A}_X(D) \cap F$$

and

$$H^1(X, \mathcal{F}(D)) \simeq \mathbb{A}_X/(\mathbb{A}_X(D) + F),$$

which is Serre's adelic interpretation of the cohomology [Ser88, Ch. II, §5].

2.2. **Differentials and residues.** The $F$-module of Kähler differentials on $X$ is the module $\Omega^1_{F/k}$, which is universal with respect to the following properties.

- **K1** There exists a $k$-linear map $d : F \to \Omega^1_{F/k}$ satisfying the Leibniz rule $d(fg) = fdg + gdf$.
- **K2** The $F$-module $\Omega^1_{F/k}$ is generated by the elements $df$, $f \in F$.

Since $X$ is an algebraic curve, $\dim_F \Omega^1_{F/k} = 1$. Let $t \in F$ be a Zariski local coordinate at point $P$ — a rational function on $X$ satisfying $v_P(t) = 1$. Then $dt$ is a generating element of the $F$-module $\Omega^1_{F/k}$, i.e., every Kähler differential can be written as $\omega = fdt$ for some $f \in F$. The order of $\omega \in \Omega^1_{F/k}$ at $P$ is defined by

$$v_P(\omega) = v_P(f).$$

The order does not depend on the choice of a local coordinate at $P$ and defines a valuation on $\Omega^1_{F/k}$.

The $O_P$-modules $\Omega^1_{O_P/k}$ for all points $P \in X$ form an algebraic coherent sheaf $\underline{\Omega}$ — a subsheaf of the constant sheaf $\underline{\Omega^1_{F/k}}$. Moreover,

$$\Omega^1_{F/k} = \Omega^1_{O_P/k} \underset{O_P}{\otimes} F.$$

In the case when the field $k$ has characteristic 0, the $F_P$-module $\Omega^1_{F_P/k}$ for every $P \in X$ is an infinite-dimensional $F_P$-vector space (the mapping $d$ is not continuous with respect to the $\mathfrak{p}$-adic topology of $F_P$). Following Serre, one defines

$$\tilde{\Omega}^1_{F_P/k} = \Omega^1_{F_P/k}/\mathcal{Q},$$

where $\mathcal{Q} = \cap_{n\geq 0}\,\mathfrak{p}^n d(\mathcal{O}_P)$, so that $\dim_{F_P} \tilde{\Omega}^1_{F_P/k} = 1$ (see [Ser88, Ch. II, §11]). The $F_P$-module $\tilde{\Omega}^1_{F_P/k}$ is the completion of the $F$-module $\Omega^1_{F/k}$ with respect to the valuation $v_P$. The completion of the $O_P$-module $\Omega^1_{O_P/k}$ is an $\mathcal{O}_P$-module $\tilde{\Omega}^1_{\mathcal{O}_P/k}$ and

$$\tilde{\Omega}^1_{F_P/k} = \tilde{\Omega}^1_{\mathcal{O}_P/k} \underset{\mathcal{O}_P}{\otimes} F_P.$$

The $\mathbb{A}_X$-module of adeles $\mathbf{\Omega}_X$ of the sheaf $\underline{\Omega}$,

$$\mathbf{\Omega}_X = \coprod_{P\in X} \tilde{\Omega}^1_{F_P/k},$$

is a restricted direct product over all points $P \in X$ of the $F_P$-modules $\tilde{\Omega}^1_{F_P/k}$ with respect to the $\mathcal{O}_P$-modules $\tilde{\Omega}^1_{\mathcal{O}_P/k}$. The $F$-module $\Omega^1_{F/k}$ is contained in all $F_P$-modules $\tilde{\Omega}^1_{F_P/k}$ and is diagonally embedded into $\mathbf{\Omega}_X$ by

$$\Omega^1_{F/k} \ni \omega \mapsto \{\omega|_P\}_{P\in X} \in \mathbf{\Omega}_X.$$

The $k$-vector space $\mathbf{\Omega}_X$ has a linear topology with the base of the neighborhoods of zero given by the subspaces $\mathbf{\Omega}_X(D)$ for all $D \in \operatorname{Div}(X)$, where

$$\mathbf{\Omega}_X(D) = \{\omega = \{\omega_P\}_{P\in X} \in \mathbf{\Omega}_X : v_P(\omega_P) \geq v_P(D) \ \text{ for all } \ P \in X\},$$

and is locally linear compact. The maps $d : F_P \to \tilde{\Omega}^1_{F_P/k}$ for all $P \in X$ give rise to the continuous map $d : \mathbb{A}_X \to \mathbf{\Omega}_X$, satisfying the Leibniz rule.

*Remark* 2.1. The $\mathbb{A}_X$-module $\mathbf{\Omega}_X$ is essentially the set of principal part systems of degree 1 on $X$ in the sense of Eichler (see [Eic66, Ch. III, §5.2]).

Let $\omega \in \tilde{\Omega}^1_{F_P/k}$, and let $t$ be a local parameter of the field $F_P$, so that $dt$ is a basis of the $F_P$-module $\tilde{\Omega}^1_{F_P/k}$. The residue map $\operatorname{Res}_P : \tilde{\Omega}^1_{F_P/k} \to k$ is defined by

$$\operatorname{Res}_P(\omega) = c_{-1}, \quad \text{where} \quad \omega = \sum_{n\gg-\infty}^{\infty} c_n t^n dt,$$

and the symbol $n \gg -\infty$ indicates that summation goes only over a finitely many negative values of $n$. The definition of the residue does not depend on the choice of a local parameter. The residue map is continuous with respect

to the $\mathfrak{p}$-adic topology on $\tilde{\Omega}^1_{F_P/k}$ and the discrete topology on $k$. The local residue maps $\mathrm{Res}_P$ give rise to the global residue map $\mathrm{Res} : \mathbf{\Omega}_X \to k$,

$$\mathrm{Res}\ \omega = \sum_{P\in X} \mathrm{Res}_P(\omega_P), \quad \omega = \{\omega_P\}_{P\in X} \in \mathbf{\Omega}_X.$$

The global residue map is well-defined, continuous, and satisfies the following fundamental property.

**Theorem 2.1** (The residue formula)**.** *For every* $\omega \in \Omega^1_{F/k}$,

$$\mathrm{Res}\ \omega = \sum_{P\in X} \mathrm{Res}_P(\omega|_P) = 0.$$

2.3. **Serre's duality and Riemann-Roch theorem.** Let

$$\Omega^1_{F/k}(D) = \Omega^1_{F/k} \cap \mathbf{\Omega}_X(D) = \{\omega \in \Omega^1_{F/k} : v_P(\omega) \geq v_P(D) \text{ for all } P \in X\}.$$

Define the residue pairing $(\ ,\ ) : \mathbf{\Omega}_X \otimes_K \mathbb{A}_X \to k$ by

$$(\omega, x) = \sum_{P\in X} \mathrm{Res}_P(x_P\omega_P), \text{ where } \omega \in \mathbf{\Omega}_X,\ x \in \mathbb{A}_X.$$

The residue pairing has the following properties.

**P1** $(\omega, x) = 0$ if $\omega \in \Omega^1_{F/k}$ and $x \in F$.

**P2** $(\omega, x) =$ if $\omega \in \mathbf{\Omega}_X(D)$ and $x \in \mathbb{A}_X(D)$.

It follows from **P1-P2** that the formula $\imath(\omega)(x) = (\omega, x)$ for every $D \in \mathrm{Div}(X)$ defines a $k$-linear map

$$\imath : \Omega^1_{F/k}(D) \to \left(\mathbb{A}_X/(\mathbb{A}_X(D) + F)\right)^\vee,$$

where $V^\vee = \mathrm{Hom}(V, k)$ is the topological dual of a $k$-vector space $V$ with the linear topology.

**Theorem 2.2** (Serre's duality)**.** *For every* $D \in \mathrm{Div}(X)$ *the mapping* $\imath$ *is an isomorphism, i.e., the finite-dimensional* $k$*-vector spaces* $\Omega^1_{F/k}(D)$ *and* $\mathbb{A}_X/(\mathbb{A}_X(D) + F)$ *are dual with respect to the residue pairing.*

**Corollary 2.3** (The strong residue theorem)**.**

(i) *An adele* $x \in \mathbb{A}_X$ *corresponds to a rational function on* $X$ *under the embedding* $F \hookrightarrow \mathbb{A}_X$ *if and only if*

$$(\omega, x) = 0 \textit{ for all } \omega \in \Omega^1_{F/k}.$$

(ii) *A differential adele* $\omega \in \mathbf{\Omega}_X$ *corresponds to a Kähler differential on* $X$ *under the embedding* $\Omega^1_{F/k} \hookrightarrow \mathbf{\Omega}_X$ *if and only if*

$$(\omega, f) = 0 \textit{ for all } f \in F.$$

*Proof.* To prove part (i), observe that by Serre's duality $x \in \mathbb{A}_X(D) + F$ for every $D \in \mathrm{Div}(X)$, and $F \cap \mathbb{A}_X(D) = 0$ for $D < 0$ gives $x \in F$. To prove part (ii), let $\omega_0 \in \Omega^1_{F/k}$, $\omega_0 \neq 0$. Setting $x = \omega/\omega_0 \in \mathbb{A}_X$ we get $0 = (\omega, f) = (f\omega_0, x)$ for all $f \in F$, so that $x \in F$ by part (i). □

*Remark* 2.2. In a slightly different form, the strong residue theorem can be found in [Eic66, Ch. III, §5.3].

For $\omega \in \Omega^1_{F/k}$ set

$$(\omega) = \sum_{P\in X} v_P(\omega)\cdot P \in \mathrm{Div}(X).$$

Since $\dim_F \Omega^1_{F/k} = 1$, divisors $(\omega)$ are linear equivalent and define the divisor class $K \in \mathrm{Cl}(X)$, called the canonical class. Combining the Riemann-Roch formula for the Euler characteristic of the divisor $D$,

$$\chi(D) = h^0(D) - h^1(D) = \deg D + 1 - g,$$

and using Serre's duality and Serre's adelic interpretation of cohomology, one gets the following result.

**Theorem 2.4** (Riemann-Roch theorem)**.** *For every $D \in \mathrm{Div}(X)$,*

$$h^0(D) - h^0(K - D) = \deg D + 1 - g.$$

Effective divisor $D$ on $X$ is called non-special if $h^0(K-D) = 0$. It follows from the Riemann-Roch theorem that effective divisor $D$ of degree $g$ is non-special if and only if $h^0(D) = 1$. Equivalently, the only rational function whose poles are contained in the effective non-special divisor of degree $g$ is a constant function.

2.4. **The tame symbol.** The group of ideles $\mathbb{J}_X$ is a group of invertible elements in $\mathbb{A}_X$. Equivalently,

$$\mathbb{J}_X = \coprod_{P\in X} F_P^*$$

— a restricted direct product of the multiplicative groups $F_P^*$ with respect to the subgroups $U_P = \mathcal{O}_P^*$ of invertible elements in $\mathcal{O}_P$. By definition,

$$a = \{a_P\}_{P\in X} \in \mathbb{J}_X \text{ if } a_P \in U_P \text{ for all but finitely many } P \in X.$$

The multiplicative group $F^*$ is embedded diagonally into the group of ideles $\mathbb{J}_X$,

$$F^* \ni f \mapsto \{f|_P\}_{P\in X} \in \mathbb{J}_X.$$

The global residue map defines the pairing $\mathrm{Res}_X : \mathbb{A}_X \otimes \mathbb{J}_X \to k$ by

$$\mathrm{Res}_X(x, a) = \sum_{P\in X} \mathrm{Res}_P \Big(x_P \frac{da_P}{a_P}\Big),$$

and by the residue theorem,

$$\mathrm{Res}_X(f, g) = 0 \quad \text{for all} \quad f \in F,\, g \in F^*. \tag{2.1}$$

The tame symbol (or Tate symbol) for the field $F_P$ is defined by

$$\tau_P(f, g) = (-1)^{mn}\frac{f^n}{g^m} \mod \mathfrak{p} \in k^*,$$

where $f, g \in F_P^*$ and $m = v_P(f), n = v_P(g)$ (see, e.g., [Ser88, Ch. III, §1.3]). It satisfies the following properties:

**T1** $\tau_P(f, g_1 g_2) = \tau_P(f, g_1)\tau_P(f, g_2)$.
**T2** $\tau_P(f, g)\tau_P(g, f) = 1$.

Since $\tau_P(f, g) = 1$ when $f, g \in U_P$, the global tame symbol

$$\tau_X(a, b) = \prod_{P\in X} \tau_P(a_P, b_P), \quad a, b \in \mathbb{J}_X$$

is a well-defined map $\tau_X : \mathbb{J}_X \times \mathbb{J}_X \to k^*$ satisfying the properties **T1**–**T2**. The classical A. Weil reciprocity law is the following statement

$$\tau_X(f, g) = 1 \quad \text{for all} \quad f, g \in F^* \tag{2.2}$$

(see [Wei40], [Ser88, Ch. III, §1.4] and [MPPR08] for the modern exposition and non-abelian generalizations.) It can be considered as non-trivial multiplicative analog of the corresponding additive result (2.1).

## 3. Differential and Integral Calculus

Starting from this section, we assume that the algebraically closed field $k$ has characteristic 0, and the algebraic curve $X$ has genus $g \geq 1$.

3.1. **Differentials of the second kind and "additive functions".** Following classical terminology, a Kähler differential $\omega \in \Omega^1_{F/k}$ is said to be of the second kind if $\operatorname{Res}_P \omega = 0$ for all $P \in X$. The $k$-vector space $\Omega^{(2\text{nd})}$ of differentials of the second kind on $X$ carries a canonical skew-symmetric bilinear form $(\ ,\ )_X$ defined as follows. For every $\omega \in \Omega^{(2\text{nd})}$ let $x = \{x_P\}_{P\in X} \in \mathbb{A}_X$ be an adele satisfying

$$d\, x_P = \omega|_P \quad \text{for all} \quad P \in X.$$

For every $P \in X$ such $x_P \in F_P$ exists, is defined up to an additive constant from $k$, and $x_P \in \mathcal{O}_P$ for all but finitely many $P \in X$. Denote $x = d^{-1}\omega$ and set

$$(\omega_1, \omega_2)_X = \sum_{P\in X} \operatorname{Res}_P(d^{-1}\omega_1\, \omega_2), \quad \omega_1, \omega_2 \in \Omega^{(2\text{nd})}.$$

The bilinear form $(\ ,\ )_X$ does not depend on the choices of additive constants in the definition of $d^{-1}$ and is skew-symmetric.

The infinite-dimensional $k$-vector space $\Omega^{(2\text{nd})}$ has a $g$-dimensional subspace $\Omega^{(1\text{st})} = \Omega^1_{F/k}(0)$ consisting of differentials of the first kind. The infinite-dimensional subspace $\Omega^{(1\text{st})} \oplus dF$ of $\Omega^{(2\text{nd})}$ is isotropic with respect to the bilinear form $(\ ,\ )_X$. Since there is no canonical choice of the complementary isotropic subspace to $\Omega^{(1\text{st})} \oplus dF$ in $\Omega^{(2\text{nd})}$, the exact sequence

$$0 \to \Omega^{(1\text{st})} \oplus dF \to \Omega^{(2\text{nd})} \to \Omega^{(2\text{nd})}/(\Omega^{(1\text{st})} \oplus dF) \to 0$$

does not split canonically. Still, the following fundamental result holds (see [Che63, Ch. VI, §8] and [Eic66, Ch. III, §§5.3-5.4]), which can be considered as an algebraic de Rham theorem (see [GH78, Ch. III, §5]).

**Theorem 3.1.**

(i) *The restriction of the bilinear form* $(\ ,\ )_X$ *to* $\Omega^{(2\mathrm{nd})}/dF$ *is non-degenerate and*
$$\dim_k \Omega^{(2\mathrm{nd})}/dF = 2g.$$

(ii) *Every choice of degree* $g$ *non-special effective divisor* $D$ *on* $X$ *defines the isomorphism*
$$\Omega^{(2\mathrm{nd})}/dF \simeq \Omega^{(2\mathrm{nd})} \cap \Omega^1_{F/k}(-2D).$$

(iii) *Let* $D = P_1 + \cdots + P_g$, *where points* $P_i \in X$, $i = 1, \dots, g$, *are all distinct, be a non-special divisor. For every choice of the uniformizers* $t_i$ *at* $P_i$, *the* $k$*-vector space* $\Omega^{(2\mathrm{nd})} \cap \Omega^1_{F/k}(-2D)$ *has the basis* $\{\theta_i, \omega_i\}_{i=1}^g$, *symplectic with respect to the bilinear from* $(\ ,\ )_X$,
$$(\theta_i, \theta_j)_X = (\omega_i, \omega_j)_X = 0, \ \ (\theta_i, \omega_j)_X = \delta_{ij}, \quad i, j = 1, \dots, g.$$
*This basis consists of differentials of the first kind* $\theta_i$ *and differentials of the second kind* $\omega_i$, *uniquely characterized by the conditions*
$$v_{P_i}(\theta_j - \delta_{ij} dt_i) > 0 \ \ \text{and} \ \ v_{P_i}\left(\omega_j - \delta_{ij} t_i^{-2} dt_i\right) > 0,$$
$i, j = 1, \dots, g$.

(iv) *The subspace* $k\cdot\omega_1 \oplus \cdots \oplus k\cdot\omega_g$ *is a complementary isotropic subspace to* $\Omega^{(1\mathrm{st})} \oplus dF$ *in* $\Omega^{(2\mathrm{nd})}$.

*Proof.* Let $(\omega)_\infty = n_1 Q_1 + \cdots + n_l Q_l$ be the polar divisor of $\omega \in \Omega^{(2\mathrm{nd})}$. Since $\operatorname{char} k = 0$, for every $Q_i$, $i = 1, \dots, l$, there exists $f_i \in F$ such that $v_{Q_i}(\omega - df_i) \geq 0$. Now define $x = \{x_P\}_{P \in X} \in \mathbb{A}_X$ by
$$x_P = \begin{cases} f_i|_{Q_i} & P = Q_i, \quad i = 1, \dots, l, \\ 0 & \text{otherwise.} \end{cases}$$
Since divisor $D$ of degree $g$ is non-special we have $\Omega^1_{F/k}(D) = \{0\}$, and by Serre duality $\mathbb{A}_X(D) + F = \mathbb{A}_X$. Thus there exists $f \in F$ with the property $v_P(f - x) \geq -v_P(D)$ for all $P \in X$, so that $(\omega - df) \geq -2D$. Since $D$ is non-special, such $f$ is unique, and this proves part (ii).

To show that $\dim_k \Omega^{(2\mathrm{nd})}/dF = 2g$ we observe that $\dim_k \Omega^1_{F/k}(-2D) = 3g - 1$ and $\dim_k \Omega^1_{F/k}(-D) = 2g - 1$, as it follows from the Riemann-Roch theorem. Denote by $\Omega^{(3\mathrm{rd})}$ the $k$-vector space of the differentials of the third kind — the subspace of $\Omega^1_{F/k}$ consisting of differentials with only simple poles. Since $\Omega^{(2\mathrm{nd})} \cap \Omega^{(3\mathrm{rd})} = \Omega^{(1\mathrm{st})}$ and $\Omega^{(3\mathrm{rd})} \cap \Omega^1_{F/k}(-2D) = \Omega^1_{F/k}(-D)$, we conclude
$$\begin{aligned} &\dim_k \Omega^{(2\mathrm{nd})} \cap \Omega^1_{F/k}(-2D) + \dim_k \Omega^1_{F/k}(-D) \\ &\quad = \dim_k \Omega^1_{F/k}(-2D) + \dim_k \Omega^{(1\mathrm{st})}, \end{aligned}$$
so that by part (ii),
$$\dim_k \Omega^{(2\mathrm{nd})}/dF = (3g - 1) - (2g - 1) + g = 2g.$$

To finish the proof, consider the $k$-linear mapping

$$L:\Omega^{(\text{2nd})}\cap\Omega^1_{F/k}(-2D)\to k^{2g},$$

defined by $L(\omega)=(\alpha_1(\omega),\dots,\alpha_g(\omega),\beta_1(\omega),\dots,\beta_g(\omega))$, where

$$v_{P_i}\big(\omega-(\alpha_i(\omega)t_i^{-2}+\beta_i(\omega)dt_i)\big)>0,\quad i=1,\dots,g.$$

Since divisor $D$ is non-special, the mapping $L$ is injective and, therefore, is an isomorphism. The differentials $\omega_i$ and $\theta_i$ are obtained by choosing the only non-zero components of $L$ to be, respectively, $\alpha_i=1$ and $\beta_i=1$. ☐

*Remark* 3.1. The choice of a non-special effective divisor $D=P_1+\dots+P_g$ on $X$ with distinct points $P_i$ and the uniformizers $t_i$ can be considered as an algebraic analog of the choice of "$a$-cycles" on a compact Riemann surface of genus $g\geq 1$. Correspondingly, differentials $\theta_i$ are analogs of the differentials of the first kind with normalized "$a$-periods", and differentials $\omega_i$ are analogs of the differentials of the second kind with second order poles, "zero $a$-periods" and normalized "$b$-periods". The symplectic property of the basis $\{\theta_i,\omega_i\}_{i=1}^g$ is an analog of the "reciprocity law for differentials of the first and the second kind" (see [Iwa93, Ch. 5, §1] and [Kra72, Ch. VI, §3]).

*Remark* 3.2. Condition that effective non-special divisor $D$ consists of $g$ distinct points is not essential. The statement of Theorem 3.1, as well as of all other results in the paper, can be easily modified to include divisors with multiple points.

A differential of the second kind $\omega$ is said to have zero $a$-periods, if

$$(\omega,\omega_i)_X=0,\quad i=1,\dots,g.$$

It follows from Theorem 3.1 that differential of the first kind with zero $a$-periods is zero. The vector space $\Omega_0^{(\text{2nd})}$ of differentials of the second kind with zero $a$-periods has the following properties.

**Proposition 3.1.**

(i) *The $k$-vector space $\Omega_0^{(\text{2nd})}$ is complementary isotropic subspace to $\Omega^{(\text{1st})}$ in $\Omega^{(\text{2nd})}$ and*

$$\Omega_0^{(\text{2nd})}=k\cdot\omega_1\oplus\dots\oplus k\cdot\omega_g\oplus dF.$$

(ii) *There is a direct sum decomposition*

$$\Omega_0^{(\text{2nd})}=\bigoplus_{P\in X}\Omega_0(*P),$$

*where $\Omega_0(*P)$is the subspace of differentials of the second kind with zero $a$-periods and only pole at $P\in X$.*

(iii) *For every $P\in X$ the $k$-vector space $\Omega_0(*P)$ has a natural filtration*

$$\{0\}=\Omega_0(-P)\subset\Omega_0(-2P)\dots\subset\Omega_0(-nP)\subset\dots,$$

*where $\Omega_0(-nP)$ is the subspace of differentials of the second kind with zero a-periods and only pole at $P$ of order not greater than $n$,*

$$\dim_k \Omega_0(-nP) = n-1.$$

(iv) *Every $\omega \in \Omega_0(-nP)$ admits a unique decomposition*

$$\omega = df + \sum_{i=1}^{g} c_i\omega_i,$$

*where $f \in H^0(X, \mathcal{F}(D+(n-1)P))$.*

*Proof.* Part (i) follows from Theorem 3.1 since divisor $D$ is non-special, and part (ii) is clear. Since $\dim_k \Omega^1_{F/k}(-nP) = n-1+g$, part (iii) follows from the decomposition

$$\Omega^1_{F/k}(-nP) = \Omega_0(-nP) \oplus \Omega^{(1\mathrm{st})}.$$

The divisor $D = P_1 + \cdots + P_g$ is non-special, so that $h^0(D+(n-1)P) = n$, and part (iv) also follows from Theorem 3.1. □

**Definition.** A space of "additive multi-valued functions on $X$" (additive functions for brevity) is the subspace $\mathcal{A}(X) \subset \mathbb{A}_X$ with the following properties.

**AF1** $F \subseteq \mathcal{A}(X)$.
**AF2** For every $a \in \mathcal{A}(X)$, $da = \omega \in \Omega^1_{F/k}$ (and hence $\omega \in \Omega^{(2\mathrm{nd})}$).
**AF3** If $a \in \mathcal{A}(X)$ satisfies $da = df$ for $f \in F$, then $a - f = c \in k$.

*Remark* 3.3. For every $\omega \in \Omega^{(2\mathrm{nd})}$ corresponding $a = \{a_P\}_{P\in X} = d^{-1}\omega$ is defined up to the choice of additive constants for every $P \in X$. Condition **AF3** ensures that for all $f \in F$ these choices are compatible with the equation $f = d^{-1}(df) + c$.

**Example 3.1.** For every non-special effective divisor $D$ of degree $g$ on $X$, $D = P_1 + \cdots + P_g$ with distinct points $P_i$, and a choice of uniformizers $t_i$ at $P_i$, there is an associated space of additive functions $\mathcal{A}(X; D)$ with "zero $a$-periods", defined as follows. Let $\eta_i \in \mathbb{A}_X$ be the solutions of the equations

$$d\eta_i = \omega_i, \quad i = 1, \dots, g,$$

with any fixed choice of additive constants at all points $P \in X$. Since the divisor $D$ is non-special, the subspaces $k \cdot \eta_1 \oplus \cdots \oplus k \cdot \eta_g$ and $F$ of the $k$-vector space $\mathbb{A}_X$ have zero intersection. Their direct sum, the subspace

$$\mathcal{A}(X; D) = k\cdot\eta_1 \oplus \cdots \oplus k\cdot\eta_g \oplus F \subset \mathbb{A}_X, \tag{3.1}$$

satisfies properties **AF1-AF3** and the mapping $d : \mathcal{A}(X; D) \to \Omega_0^{(2\mathrm{nd})}$ is surjective. Indeed, according to Proposition 3.1, every $\omega \in \Omega_0^{(2\mathrm{nd})}$ admits a unique decomposition

$$\omega = df + \sum_{i=1}^{g} c_i\omega_i, \tag{3.2}$$

and

$$a = d^{-1}\omega = f + \sum_{i=1}^{g} c_i\eta_i + c \in \mathcal{A}(X; D). \tag{3.3}$$

*Remark* 3.4. Additive functions $a = d^{-1}\omega \in \mathcal{A}(X, D)$ are algebraic analogs of abelian integrals of the second kind with zero $a$-periods on a compact Riemann surface of genus $g$ (see, e.g., [Iwa93, Ch. V, §2]), and we can define

$$\int_P^Q \omega = a(Q) - a(P),$$

where $a(P) = a_P \bmod \mathfrak{p} \in k$ for every $P \in X$.

It is quite remarkable that using additive functions introduced in Example 3.1, one can naturally define uniformizers $t_P$ at all points $P \in X$. These uniformizers are uniquely determined by the following data: a choice of a non-special divisor $D = P_1 + \cdots + P_g$ with distinct points, uniformizers $t_i$ at $P_i$, and additive functions $\eta_1, \dots, \eta_g$. Namely, for every $P \in X$ let $\omega_P^{(2)} \in \Omega_0(-2P)$ be a differential of the second kind with the second order pole at $P$ and zero $a$-periods, uniquely characterized by the condition

$$\sum_{i=1}^{g} \big(\theta_i, \omega_P^{(2)}\big)_X = 1. \tag{3.4}$$

In particular, $\omega_{P_i}^{(2)} = \omega_i$ for $i = 1, \dots, g$. Let $\eta_P = d^{-1}\omega_P^{(2)} \in \mathcal{A}(X; D)$ be the additive function with the only simple pole at $P \in X$. According to (3.3), $\eta_P$ is defined up to an overall additive constant, which we fix by the condition that the sum of constant terms of $\eta_P|_{P_i} \in k((t_i))$ over all $i = 1, \dots, g$ is equal to zero. In particular, $\eta_{P_i} = \eta_i + c_i$ for some $c_i \in k$. For every $P \in X$ the uniformizer $t_P$ is defined by

$$t_P = -\left.\frac{1}{\eta_P}\right|_P,$$

and for $\omega_P^{(2)} = d\eta_P$ we get

$$\left.\omega_P^{(2)}\right|_P = t_P^{-2} dt_P, \quad P \in X.$$

Extending this construction, for every $P \in X$ we choose a basis $\{\omega_P^{(n+1)}\}_{n=1}^{\infty}$ of the subspace $\Omega_0(*P)$ which consists of differentials of the second kind with the only pole at $P$ of order $n+1$ and zero $a$-periods, where differentials $\omega_P^{(2)}$ are specified by (3.4). Let $\eta_P^{(n)} = d^{-1}\omega_P^{(n+1)} \in \mathcal{A}(X; D)$ be the additive function with the only pole at $P \in X$ of order $n$, where the overall additive constant in (3.3) is fixed as follows. We set $\eta_P^{(1)} = \eta_P$, and for $\eta_P^{(n)}$ with $n > 1$ we impose a condition that the constant term of $\left.\eta_P^{(n)}\right|_P \in k((t_P))$ is

zero. Introducing for every $P \in X$ the subspace $\mathcal{A}_P(X, D)$ — the $k$-span of $\eta_P^{(n)}$, $n \in \mathbb{N}$ — we have the decomposition

$$\mathcal{A}(X, D) = \left(\bigoplus_{P \in X} \mathcal{A}_P(X, D)\right) \oplus k. \tag{3.5}$$

The property that $\Omega_0^{(2\mathrm{nd})} = d\,\mathcal{A}(X; D)$ is the isotropic subspace, and the condition **AF3** can be equivalently stated as follows.

**Lemma 3.1.**

(i) *For every $P, Q \in X$ and $m, n \in \mathbb{N}$,*

$$\mathrm{Res}_P(\eta_P^{(m)} d\,\eta_Q^{(n)}) = \mathrm{Res}_Q(\eta_Q^{(n)} d\,\eta_P^{(m)}).$$

(ii) *Every $f \in F$ admits a unique "partial fraction expansion"*

$$f = \sum_{i=1}^{l} \sum_{j=1}^{n_i} c_{ij} \eta_{Q_i}^{(j)} + c,$$

*where $n_1 Q_1 + \dots n_l Q_l = (f)_\infty$ is the polar divisor of $f$, and $c, c_{ij} \in k$.*

*Proof.* Since $\mathrm{Res}_Q(da) = 0$ for every $a \in F_P$, we get for $P \neq Q$,

$$\begin{aligned} 0 = (\omega_P^{(m+1)}, \omega_Q^{(n+1)})_X &= \mathrm{Res}_P(\eta_P^{(m)} d\eta_Q^{(n)}) + \mathrm{Res}_Q(\eta_P^{(m)} d\eta_Q^{(n)}) \\ &= \mathrm{Res}_P(\eta_P^{(m)} d\eta_Q^{(n)}) - \mathrm{Res}_Q(\eta_Q^{(n)} d\eta_P^{(m)}). \end{aligned}$$

For $P = Q$ we get $0 = (\omega_P^{(m+1)}, \omega_P^{(n+1)})_X = \mathrm{Res}_P(\eta_P^{(m)} d\eta_P^{(n)})$ for all $m, n \in \mathbb{N}$. Part (ii) immediately follows from **AF3** since there are $c_{ij} \in k$ such that

$$df - \sum_{i=1}^{l} \sum_{j=1}^{n_i} c_{ij} \omega_{Q_i}^{(j+1)} \in \Omega_0^{(2\mathrm{nd})} \cap \Omega^{(1\mathrm{st})} = \{0\}. \qquad \square$$

*Remark* 3.5. The first statement of Lemma 3.1 is an algebraic analog of the classical "reciprocity law for the differentials of the second kind with zero $a$-periods" on a compact Riemann surface (see, e.g., [Iwa93, Ch. V, §1] and [Kra72, Ch. VI, §3]).

*Remark* 3.6. In the genus zero case $X = \mathbb{P}_k^1 = k \cup \{\infty\}$, $F = k(z)$, and

$$\omega_P^{(n+1)} = \frac{dz}{(z-P)^{n+1}} \text{ for } P \in k, \quad \omega_P^{(n+1)} = -z^{n-1} dz \text{ for } P = \infty.$$

Correspondingly,

$$\eta_P^{(n)}(z) = -\frac{1}{n(z-P)^n} \text{ for } P \in k, \quad \eta_P^{(n)}(z) = -\frac{z^n}{n} \text{ for } P = \infty.$$

3.2. **Differentials of the third kind and "multiplicative functions".** The $k$-vector space $\Omega^{(3rd)}$ of differentials of the third kind contains a $g$-dimensional subspace $\Omega^{(1st)}$ and the subspace $d\log F^*$ consisting of logarithmic differentials

$$d\log f = \frac{df}{f}, \quad f \in F^* = F \setminus \{0\}.$$

Let $D = P_1 + \cdots + P_g$ be a non-special divisor of degree $g$ on $X$ with distinct points, and let $\mathcal{A}(X; D)$ be the vector space of additive functions with zero $a$-periods, defined in Example 3.1.

**Definition.** Differential of the third kind $\omega$ has zero $a$-periods if

$$(\omega_i, \omega)_X = \sum_{P \in X} \operatorname{Res}_P(\eta_i\, \omega) = 0 \quad \text{for} \quad i = 1, \dots, g,$$

where $\eta_i = d^{-1}\omega_i \in \mathcal{A}(X; D)$.

It follows from (3.1) and the residue theorem that every differential of the third kind $\omega$ with zero $a$-periods satisfies

$$\sum_{P \in X} \operatorname{Res}_P(\eta_P\, \omega|_P) = 0 \tag{3.6}$$

for all $\eta \in \mathcal{A}(X; D)$.

By the Riemann-Roch theorem $\dim_k \Omega^1_{F/k}(-P - Q) = g + 1$, so that for every $P, Q \in X$, $P \neq Q$, there is a differential of the third kind with the only poles at $P$ with residue 1 and at $Q$ with residue $-1$. Such differentials form an affine space over the vector space $\Omega^{(1st)}$ of differentials of the first kind, and there exists a unique differential of the third kind with zero $a$-periods, which we denote by $\omega_{PQ}$. The differentials $\omega_{PQ}$ for all $(P, Q) \in X \times X$, $P \neq Q$, span the $k$-vector space $\Omega_0^{(3rd)}$ of differentials of the third kind with zero $a$-periods, the complementary subspace to $\Omega^{(1st)}$ in $\Omega^{(3rd)}$,

$$\Omega^{(3rd)} = \Omega_0^{(3rd)} \oplus \Omega^{(1st)}.$$

Note that so far we did not specify the choice of arbitrary constants in the definition of the additive functions $\eta_i$, $i = 1, \dots, g$, so that we cannot guarantee that logarithmic differentials $d\log f$, $f \in F^*$, have zero $a$-periods. We have the following result.

**Lemma 3.2.** *There is a choice of additive constants in the definition of* $\eta_i \in \mathcal{A}(X; D)$, $i = 1, \dots, g$, *such that* $d\log F^* \subset \Omega_0^{(3rd)}$, *and all such choices are parametrized by* $g$ *elements in* $\operatorname{Hom}(\operatorname{Pic}_0(X), k)$.

*Proof.* Every choice of additive constants for $\eta_i$ — an element $c_i = \{c_{iP}\}_{P \in X} \in \mathbb{A}_X$ — defines a homomorphism $\pi_i : \operatorname{Div}_0(X) \to k$ by

$$\pi_i(D) = \sum_j c_{iQ_j} n_j, \quad \text{where} \quad D = \sum_j n_j Q_j.$$

The condition

$$\sum_{P\in X}\mathrm{Res}_P(\eta_i\,d\log f)=0\quad\text{for all}\quad f\in F^*$$

implies that restriction of $\pi_i$ to the subgroup $\mathrm{PDiv}(X)$ of principal divisors is given by

$$\pi_i((f))=\mathrm{Res}_{P_i}\frac{d\log f}{t_i},\quad i=1,\dots,g.$$

Since the field $k$ has characteristic zero, it is an injective $\mathbb{Z}$-module and, therefore, $\mathrm{Ext}^1(A,k)=0$ for any $\mathbb{Z}$-module $A$. Applying this to the $\mathbb{Z}$-module $A=\mathrm{Pic}_0(X)=\mathrm{Div}_0(X)/\mathrm{PDiv}(X)$, we see that the restriction mapping

$$\mathrm{Hom}(\mathrm{Div}_0(X),k)\to\mathrm{Hom}(\mathrm{PDiv}(X),k)$$

is surjective. □

**Definition.** The space of additive functions $\mathcal{A}(X;D)$ is said to be compatible with the multiplicative group $F^*$ of the field $F$ of rational functions on $X$, if $d\log F^*\subset\Omega_0^{(3\mathrm{rd})}$.

In this case, denoting by $(f)=\sum_{i=1}^n(Q_i-R_i)$ the divisor of $f\in F^*$, we have

$$\frac{df}{f}=\sum_{i=1}^n\omega_{Q_iR_i}. \tag{3.7}$$

Indeed, $d\log f-\sum_{i=1}^n\omega_{Q_iR_i}=0$ since it is a differential of the first kind with zero $a$-periods.

For every $\omega_{QR}\in\Omega_0^{(3\mathrm{rd})}$ let $f_{QR}=\{f_{QR,P}\}_{P\in X}\in\mathbb{J}_X$ be an idele such that

$$\omega_{QR}=d\log f_{QR}=\frac{df_{QR}}{f_{QR}}.$$

It has the property $v_P(f_{QR,P})=0$ for all $P\neq Q,R$, $v_P(f_{QR,P})=1$ for $P=Q$, and $v_P(f_{QR,P})=-1$ for $P=R$. For every $P\in X$, the element $f_{QR,P}\in F_P^*$ is defined up to an arbitrary multiplicative constant $c_P\in k^*$. Since for arbitrary distinct points $Q,R,S\in X$

$$\omega_{QR}+\omega_{RS}=\omega_{QS},$$

we get that for every $P\in X$ and arbitrary choice of multiplicative constants,

$$\frac{f_{QR,P}f_{RS,P}}{f_{QS,P}}=c_{QRS,P}\in k^*. \tag{3.8}$$

Moreover, equation (3.8) extends to the case of coincident points $Q,R,S$ if we put $f_{QQ,P}=\alpha_{QQ,P}\in k^*$, etc. It follows from (3.8) that for every $P\in X$ the elements $c=\{c_{QRS,P}\}$ satisfy the “3-cocycle condition”

$$c_{QRS,P}\,c_{QST,P}=c_{QRT,P}\,c_{RST,P}.$$

Clearly, the 3-cocycle $c$ is a coboundary: there exists a "2-cochain" $b = \{b_{QR,P}\}$ such that

$$c_{QRS,P} = \frac{b_{QR,P}\, b_{RS,P}}{b_{QS,P}}.$$

This shows that we can choose multiplicative constants in the definition of $f_{QR,P}$ such that equation (3.8) becomes

$$\frac{f_{QR,P} f_{RS,P}}{f_{QS,P}} = 1. \tag{3.9}$$

Now it follows from (3.9) that for every $Q \in X$ there is an idele $e_Q = \{e_{Q,P}\}_{P\in X} \in \mathbb{J}_X$ such that $v_P(e_{Q,P}) = 0$ for all $P \neq Q$, $v_P(e_{Q,P}) = 1$ for $P = Q$, and

$$f_{QR} = \frac{e_Q}{e_R}.$$

The elements $e_{Q,P} \in F_P^*$ for every $P \in X$ are defined up to multiplicative constants, and we chose them in such a way that $c_{Q,P} = e_{Q,P} \bmod \mathfrak{p} \in k^*$, defined for $P \neq Q$, always satisfy $c_{Q,P} = -c_{P,Q}$. We will use convenient notation $c_{Q,P} = e_Q(P)$. For $P = Q$ we have $e_{Q,Q} = c_{Q,Q} t_Q(1 + O(t_Q))$, where $c_{Q,Q} \in k^*$, and $t_Q$ are the uniformizers at points $Q \in X$ associated with the space of additive functions $\mathcal{A}(X, D)$, defined in Section 3.1. We will also use the notation $c_{Q,Q} = \dot{e}_Q(Q)$.

For distinct points $P, Q, R, S \in X$ we define an algebraic analog of an exponential of the abelian integral of the third kind on a compact Riemann surface of genus $g$ by

$$\exp\int_P^Q \omega_{RS} = \frac{f_{RS}(Q)}{f_{RS}(P)},$$

where $f_{RS}(P) = f_{RS,P} \bmod \mathfrak{p} \in k^*$, $f_{RS}(P) = \dfrac{e_R(P)}{e_S(P)}$. The following result is an analog of the reciprocity law for the normalized differentials of the third kind — classical "exchange law of variable and parameter" (see, e.g., [Iwa93, Ch. V, §1] and [Kra72, Ch. VI, §3]).

**Lemma 3.3.** *For distinct points $P, Q, R, S \in X$,*

$$\exp\int_R^S \omega_{PQ} = \exp\int_P^Q \omega_{RS}.$$

*Proof.* We have

$$\begin{aligned}\exp\int_R^S \omega_{PQ} &= \frac{f_{PQ}(S)}{f_{PQ}(R)} = \frac{e_P(S)e_Q(R)}{e_Q(S)e_P(R)}\\ &= \frac{e_S(P)e_R(Q)}{e_S(Q)e_R(P)} = \frac{f_{RS}(Q)}{f_{RS}(P)} = \exp\int_P^Q \omega_{RS}.\end{aligned}$$

□

It follows from (3.7) that for $f \in F^*$

$$f|_P = c_P \prod_{i=1}^n f_{Q_iR_i,P} = c_P \prod_{i=1}^n \frac{e_{Q_i,P}}{e_{R_i,P}}$$

for every $P \in X$, where $(f) = \sum_{i=1}^{n}(Q_i - R_i)$. We finalize our choice of multiplicative constants $c_{P,Q}$ by the condition that $c_P = c \in k^*$ — a constant depending on $f$ — for all $P \in X$, so that every $f \in F^*$ can be written in a 'factorized form'

$$f = c \prod_{i=1}^{n} f_{Q_i R_i}. \tag{3.10}$$

**Proposition 3.2.** *There is a choice of constants $\{c_{P,Q}\}_{P,Q \in X} \in k^*$ in the definition of the ideles $e_P \in \mathbb{J}_X$, $P \in X$, satisfying $c_{P,Q} = -c_{Q,P}$ for all $P \neq Q$, and having the property that the factorization formula (3.10) holds for every $f \in F^*$.*

*Proof.* For $u \in \mathcal{O}_P^*$ put $u(P) = u \bmod \mathfrak{p} \in k^*$. We need to show that there exist $c_{P,Q} \in k^*$ satisfying $c_{P,Q} = -c_{Q,P}$ for $P \neq Q$, such that for every $f \in F^*$,

$$\prod_{i=1}^{n} \frac{c_{Q_i,P}}{c_{R_i,P}} = c \left(\frac{f}{t_P^{v_P(f)}}\right)(P)$$

for all $P \in X$ and some $c = c(f) \in k^*$, where $(f) = \sum_{i=1}^{n}(Q_i - R_i)$.

For every $D \in \mathrm{PDiv}(X)$ choose some $f \in F^*$ such that $D = (f)$, put

$$c_1(D, P) = \left(\frac{f}{t_P^{v_P(f)}}\right)(P), \quad P \in X,$$

and extend it to the group homomorphism $c_1 : \mathrm{PDiv}(X) \times \mathrm{Div}(X) \to k^*$ by multiplicativity. Similarly, define $c_2 : \mathrm{Div}(X) \times \mathrm{PDiv}(X) \to k^*$ by

$$c_2(P, D) = (-1)^{v_P(f)} c_1(D, P), \quad D \in \mathrm{PDiv}(X).$$

We claim that

$$c_1|_{\mathrm{PDiv}(X)\times\mathrm{PDiv}(X)} = c_2|_{\mathrm{PDiv}(X)\times\mathrm{PDiv}(X)}. \tag{3.11}$$

Indeed,

$$c_1((f),(g)) = \prod_{P \in (g)} \left(\frac{f}{t_P^{v_P(f)}}\right)(P) = \prod_{P \in X} \left(\frac{f^{v_P(g)}}{t_P^{v_P(f)v_P(g)}}\right)(P)$$

and

$$c_2((f),(g)) = (-1)^{\sum_{P\in X} v_P(f)v_P(g)} c_1((g),(f)).$$

As the result, equation (3.11) takes the form

$$c_1((f),(g)) = (-1)^{\sum_{P\in X} v_P(f)v_P(g)} c_1((g),(f)),$$

which is A. Weil reciprocity law (2.2).

Now it follows from equation (3.11) that there is a group homomorphism $c : \mathrm{Div}(X) \times \mathrm{Div}(X) \to k^*$ satisfying

$$c(D_1, D_2) = (-1)^{\deg D_1 \deg D_2 + \sum_{P\in X} v_P(D_1)v_P(D_2)} c(D_2, D_1)$$

for all $D_1, D_2 \in \mathrm{Div}(X)$, and such that its restrictions to the subgroups $\mathrm{PDiv}(X) \times \mathrm{Div}(X)$ and $\mathrm{Div}(X) \times \mathrm{PDiv}(X)$ coincide, respectively, with $c_1$ and $c_2$. The homomorphism $c$ necessarily has the form

$$c(D_1, D_2) = \prod_{i,j} c(Q_i, R_j)^{n_i m_j}, \quad \text{where} \quad D_1 = \sum_i n_i Q_i,\ D_2 = \sum_j m_j R_j,$$

and the constants $c_{P,Q} = c(P, Q)$ satisfy the required conditions. □

*Remark* 3.7. The family of ideles $e_P \in \mathbb{J}_X$, $P \in X$, has the property

$$\omega_{PQ} = d(\log e_P - \log e_Q), \tag{3.12}$$

and can be considered as an algebraic analog of the classical Schottky-Klein prime form — a special multi-valued function $E(x, y)$ on the complex surface $X \times X$ with a single simple pole along the diagonal $\Delta$, which satisfies (3.12). We refer to [Fay73, Mum07] for the analytic definition of $E(x, y)$, and to [Rai89] for the algebraic definition. In the complex analytic case,

$$\omega_B = d_x d_y \log E(x, y) \tag{3.13}$$

is the so-called Bergmann kernel — a symmetric bidifferential on $X \times X$ with a single second-order pole on the diagonal $\Delta$ with the biresidue 1 and zero $a$-periods. As in [BR96], one can show that there is an algebraic analog of the Bergmann kernel for the algebraic curve $X$. It would be also interesting to introduce an analog of the Schottky-Klein prime form starting from (3.13). This approach would require using Parshin's adeles [Par76] for the algebraic surface $X \times X$, and is beyond the scope of this paper.

*Remark* 3.8. In the genus zero case $X = \mathbb{P}^1_k$, $F = k(z)$, and the family of ideles $e_P \in \mathbb{J}_X$ is given explicitly by

$$e_{P,Q} = \begin{cases} z - P \in F \subset F_Q = k((z - Q)) & \text{for } P, Q \in k, \\ 1 - z^{-1}P \in F \subset F_\infty = k((z^{-1})) & \text{for } P \in k,\ Q = \infty, \\ 1 \in F \subset F_Q = k((z - Q)) & \text{for } P = \infty,\ Q \in k, \\ z^{-1} \in F \subset F_\infty = k((z^{-1})) & \text{for } P = Q = \infty. \end{cases}$$

Correspondingly, $c_{P,Q} = Q - P$ for $P, Q \in k$, $P \neq Q$, $c_{\infty,P} = -c_{P,\infty} = 1$ for $P \in k$, and $c_{P,P} = 1$ for all $P \in k \cup \{\infty\}$.

Similar to the previous section, we define algebraic analogs of multiplicative multi-valued functions on a compact Riemann surface (see, e.g., [Iwa93, Ch. 5, §2] and [Kra72, Ch. VI, §4]) as follows.

**Definition.** A group of "multiplicative multi-valued functions on $X$" (multiplicative functions for brevity) is a sugroup $\mathcal{M}(X) \subset \mathbb{J}_X$ with the following properties.

**MF1** $F^* \subseteq \mathcal{M}(X)$.

**MF2** For every $m \in \mathcal{M}(X)$, $\dfrac{dm}{m} = \omega \in \Omega^1_{F/k}$ (and hence $\omega \in \Omega^{(3\mathrm{rd})}$).

**MF3** If $m \in \mathcal{M}(X)$ and $f \in F^*$ satisfy $\dfrac{dm}{m} = \dfrac{df}{f}$, then $m = cf$, $c \in k^*$.

**Example 3.2.** Let $D = P_1 + \cdots + P_g$ be a non-special divisor with distinct points, and let $\mathcal{A}(X, D)$ be the corresponding space of additive functions compatible with the multiplicative group $F^*$. We define the associated group of multiplicative functions $\mathcal{M}(X, D)$ as the subgroup of $\mathbb{J}_X$ generated by the ideles

$$f_{PQ} = \frac{e_P}{e_Q}, \quad P \neq Q \in X.$$

Properties **MF1**–**MF3** immediately follow from our definition of the family $e_P \in \mathbb{J}_X$, $P \in X$. The mapping

$$\mathrm{Div}_0(X) \ni D = \sum_{i=1}^{n}(Q_i - R_i) \mapsto m_D = \prod_{i=1}^{n} f_{Q_i R_i} \in \mathcal{M}(X, D)$$

establishes the group isomorphism $\mathrm{Div}_0(X) \simeq \mathcal{M}(X, D)/k^*$. It follows from the Riemann-Roch theorem that every $D \in \mathrm{Div}_0(X)$ can be represented as

$$D = (f) + \sum_{i=1}^{g}(Q_i - P_i), \quad f \in F,$$

and such representation is unique if and only if the divisor $Q_1 + \cdots + Q_g$ is non-special. Thus for every $m = cm_D \in \mathcal{M}(X, D)$ we have the decomposition $m = cf\prod_{i=1}^{g} f_{Q_i P_i}$.

*Remark* 3.9. In the genus zero case $f_{PQ} \in F = k(z)$ are given explicitly by

$$f_{PQ} = \frac{z - P}{z - Q} \text{ for } P, Q \in k, \quad f_{PQ} = \frac{1}{z - Q} \text{ for } P = \infty, \, Q \in k.$$

For every $P \in X$ put

$$F_P^- = \mathcal{A}_P(X, D)|_P \subset F_P,$$

and let $u_P^{(n)}$, $n \in \mathbb{N}$, be the basis in $\mathfrak{p} \subset F_P$ dual to the basis $v_P^{(n)} = \eta_P^{(n)}\Big|_P$ in $F_P^-$ with respect to the pairing $c : \mathfrak{p} \otimes F_P^- \to k$ given by

$$c(u, v) = -\operatorname{Res}_P(udv).$$

**Lemma 3.4.** *Multiplicative functions* $f_{PQ} = \{f_{PQ,R}\}_{R \in X} \in \mathcal{M}(X, D)$ *in Example* 3.2 are given by

$$f_{PQ,R} = \begin{cases} \dfrac{e_P(R)}{e_Q(R)} \exp\left\{\displaystyle\sum_{n=1}^{\infty} u_R^{(n)}\left(\eta_R^{(n)}(Q) - \eta_R^{(n)}(P)\right)\right\} & \text{if} \quad R \neq P, Q, \\ \dfrac{\dot{e}_P(P)}{e_Q(P)} t_P \exp\left\{\displaystyle\sum_{n=1}^{\infty} u_P^{(n)} \eta_P^{(n)}(Q)\right\} & \text{if} \quad R = P, \\ \dfrac{e_P(Q)}{\dot{e}_Q(Q)} t_Q^{-1} \exp\left\{-\displaystyle\sum_{n=1}^{\infty} u_Q^{(n)} \eta_Q^{(n)}(P)\right\} & \text{if} \quad R = Q. \end{cases}$$

Here for $R \neq P$ we put $\eta_R^{(n)}(P) = \eta_R^{(n)}\Big|_P \bmod \mathfrak{p} \in k$, etc.

*Proof.* Let $\mathfrak{r}$ be the prime ideal of the valuation ring $\mathcal{O}_R$. For $R \neq P, Q$ we have

$$f_{PQ,R} = \frac{e_P(R)}{e_Q(R)} \exp g_R, \quad \text{where} \quad g_R = \sum_{n=1}^{\infty} a_n u_R^{(n)} \in \mathfrak{r}.$$

It follows from (3.6) that

$$0 = (\omega_R^{(n)}, \omega_{PQ})_X = \operatorname{Res}_R(\eta_R^{(n)} dg_R) + \operatorname{Res}_P(\eta_R^{(n)} \omega_{PQ}) + \operatorname{Res}_Q(\eta_R^{(n)} \omega_{PQ})$$

and

$$a_n = c(g_R, \eta_R^{(n)}) = -\operatorname{Res}_R(g_R\, d\eta_R^{(n)}) = \operatorname{Res}_R(\eta_R^{(n)} dg_R) = -\eta_R^{(n)}(P) + \eta_R^{(n)}(Q).$$

For $R = P$ we have

$$f_{PQ,P} = \frac{\dot{e}_P(P)}{e_Q(P)}\, t_P \exp g_P, \quad \text{where} \quad g_P = \sum_{n=1}^{\infty} a_n u_P^{(n)} \in \mathfrak{p}.$$

It follows from (3.6), and condition that the constant term in the expansion of $\eta_P^{(n)}$ at $P$ with respect to $t_P$ is zero, that

$$\begin{aligned}0 = (\omega_P^{(n)}, \omega_{PQ})_X &= \operatorname{Res}_P\left(\eta_P^{(n)} \frac{dt_P}{t_P}\right) + \operatorname{Res}_P(\eta_P^{(n)} dg_P) + \operatorname{Res}_Q(\eta_P^{(n)} \omega_{PQ}) \\ &= -\operatorname{Res}_P(g_P d\eta_P^{(n)}) - \eta_P^{(n)}(Q),\end{aligned}$$

which gives $a_n = \eta_P^{(n)}(Q)$. The case $R = Q$ is considered similarly. $\square$

*Remark* 3.10. In the genus zero case $F = k(z)$ and $u_P^{(n)} = -(z-P)^n$ for $P \in k$, $u_\infty^{(n)} = -z^{-n}$.

**Proposition 3.3.** *Let $\mathcal{M}(X, D) \subset \mathbb{J}_X$ be the group of multiplicative functions, defined in Example 3.2.* The restriction of the global tame symbol $\tau_X$ to $\mathcal{M}(X, D) \times \mathcal{M}(X, D)$ is the identity map.

*Proof.* It is sufficient to show that $\tau(f_{PQ}, f_{RS}) = 1$ for all $P, Q, R, S \in X$ such that $P \neq Q$ and $R \neq S$. Indeed, when all points $P, Q, R, S$ are distinct, we have

$$\begin{aligned}\tau_X(f_{PQ}, f_{RS}) &= \tau_P(f_{PQ}, f_{RS})\tau_Q(f_{PQ}, f_{RS})\tau_R(f_{PQ}, f_{RS})\tau_S(f_{PQ}, f_{RS}) \\ &= \frac{f_{RS}(Q) f_{PQ}(R)}{f_{RS}(P) f_{PQ}(S)} = 1\end{aligned}$$

as in the proof of Lemma 3.3. Similarly,

$$\begin{aligned}\tau_X(f_{PQ}, f_{PS}) &= \tau_P(f_{PQ}, f_{PS})\tau_Q(f_{PQ}, f_{PS})\tau_S(f_{PQ}, f_{PS}) \\ &= -\frac{f_{PQ}}{f_{PS}}(P) \frac{f_{PS}(Q)}{f_{PQ}(S)} \\ &= -\frac{e_S(P) e_P(Q) e_Q(S)}{e_Q(P) e_S(Q) e_P(S)} = (-1)^4 = 1\end{aligned}$$

since $e_P(Q) = -e_Q(P)$ for all $P \neq Q$. Finally,

$$\tau_X(f_{PQ}, f_{PQ}) = \tau_P(f_{PQ}, f_{PQ})\tau_Q(f_{PQ}, f_{PQ}) = (-1)^2 = 1. \qquad \square$$

*Remark* 3.11. Proposition 3.3 can be considered as generalized A. Weil reciprocity law for multiplicative functions.

## 4. Local Theory

Let $K$ be a complete closed field — a complete discrete valuation field with the valuation ring $\mathcal{O}_K$, the maximal ideal $\mathfrak{p}$, and the algebraically closed residue field $k = \mathcal{O}_K/\mathfrak{p}$. Every choice of the uniformizer defines an isomorphism $K \simeq k((t))$, so that $K$ can be interpreted as a "geometric loop algebra" over the field $k$. The field $K = F_P$, where $P$ is a point on an algebraic curve $X$ over $k$, will be our main example.

In this section we introduce infinite-dimensional algebras and groups naturally associated with the field $K$, and construct their highest weight modules. For the case $K = F_P$ these objects would define local quantum field theories at $P \in X$. Specifically, we consider the following local QFT's.

1. "QFT of additive bosons", which corresponds to the Heisenberg Lie algebra $\mathfrak{g}$ — a one-dimensional central extension of the geometric loop algebra $\mathfrak{gl}_1(K) = K$.
2. "QFT of lattice bosons", which corresponds to the lattice Lie algebra $\mathfrak{l}$ associated with the Heisenberg Lie algebra $\mathfrak{g}$ and the lattice $\mathbb{Z}$.
3. "QFT of multiplicative bosons", which corresponds to the pair $(G, \mathfrak{g})$, where $G$ is a central extension of the abelian group $\mathrm{GL}_1(K) = K^*$ by the tame symbol.

### 4.1. **The Heisenberg algebra.**

Let $\Omega^1_{K/k}$ be the $K$-module of Kähler differentials, and let $\tilde{\Omega}^1_{K/k} = \Omega^1_{K/k}/\mathcal{Q}$, where $\mathcal{Q} = \cap_{n\geq 0}\,\mathfrak{p}^n d(\mathcal{O})$ (see Section 2.2). The abelian Lie algebra $\mathfrak{gl}_1(K) = K$ over the field $k$ is equipped with the bilinear, skew-symmetric form $c : \wedge^2 K \to k$,

$$c(f, g) = -\operatorname{Res}(f dg), \quad f, g \in K,$$

where $dg \in \tilde{\Omega}^1_{K/k}$. Bilinear form $c$ is continuous with respect to the $\mathfrak{p}$-adic topology in $K$ and the discrete topology in $k$, i.e., $c \in H^2_c(K, k) \simeq \mathrm{Hom}_c(\wedge^2 K, k)$ — the group of continuous 2-cocycles of $K$ with values in $k$.

**Definition.** The Heisenberg Lie algebra $\mathfrak{g}$ is a one-dimensional central extension of $K$

$$0 \to k \cdot C \to \mathfrak{g} \to K \to 0$$

with the 2-cocycle $c$.

Denoting by $[\ ,\ ]$ the Lie bracket in $\mathfrak{g} = K \oplus k \cdot C$, we get

$$[f + a\,C, g + b\,C] = c(f, g)\,C, \quad f, g \in K,\ a, b \in k.$$

The Lie subalgebra $\mathfrak{g}_+ = \mathcal{O}_K \oplus k \cdot C$ is a maximal abelian subalgebra in $\mathfrak{g}$.

*Remark* 4.1. Let $\operatorname{Aut}\mathcal{O} = \{u \in \mathcal{O} : v(u) = 1\}$ be the group of continuous automorphisms of the valuation ring $\mathcal{O} = k[[t]]$ (see [FBZ04]). It is easy to show that every continuous linear map $l : k((t)) \otimes_k k((t)) \to k$ which satisfies

$$l(f \circ u, g \circ u) = l(f, g)$$

for all $f, g \in k((t))$ and $u \in \operatorname{Aut}\mathcal{O}$, is a constant multiple of the map $c$. This clarifies the natural role of the 2-cocycle $c$ of $K$. In particular, every $\operatorname{Aut}\mathcal{O}$-invariant bilinear form $l$ is necessarily skew-symmetric, which can be considered as a simple algebraic version of the "spin-statistics theorem".

**Definition.** A $\mathfrak{g}$-module is a $k$-vector $V$, with a discrete topology, equipped with a $k$-algebra homomorphism $\rho : \mathfrak{g} \to \operatorname{End} V$ such that the $\mathfrak{g}$-action on $V$ is continuous, and $\rho(C) = \boldsymbol{I}$ — the identity endomorphism of $V$.

Equivalently, for every $v \in V$ there exists a open subspace $U$ in $K$, commensurable with $\mathfrak{p}$, that annihilate $v$: $\rho(U)\, v = 0$. Setting $\boldsymbol{f} = \rho(f) \in \operatorname{End} V$ for every $f \in K$, we get

$$[\boldsymbol{f}, \boldsymbol{g}] = c(f, g)\boldsymbol{I}$$

—projective representation of the abelian Lie algebra $K$.

*Remark* 4.2. Every choice of the uniformizer defines an isomorphism $K \simeq k((t))$ and a basis basis $\{t^n\}_{n\in\mathbb{Z}}$ for $K$. Denoting $\boldsymbol{\alpha}_n = \rho(t^n)$ and using $c(t^m, t^n) = m\delta_{m,-n}$, we get commutation relations of the "oscillator algebra"

$$[\boldsymbol{\alpha}_m, \boldsymbol{\alpha}_n] = m\delta_{m,-n}\boldsymbol{I},$$

that characterizes free bosons in two-dimensional QFT.

**Definition.** The highest weight module for the Heisenberg Lie algebra $\mathfrak{g}$ is an irreducible $\mathfrak{g}$-module with the vector $\mathbf{1} \in V$ annihilated by the abelian subalgebra $\mathcal{O}_K \oplus \{0\}$.

The following result is well-known (see, e.g., [Kac90, Lemma 9.13]).

**Theorem 4.1.** *All irreducible highest weight modules for the Heisenberg Lie algebra $\mathfrak{g}$ are the trivial one-dimensional module $k = k \cdot \mathbf{1}$ with the highest vector $\mathbf{1} = 1 \in k$, and the Fock module*

$$\mathscr{F} = \operatorname{Ind}_{\mathfrak{g}_+}^{\mathfrak{g}} k,$$

*induced from the one-dimensional $\mathfrak{g}_+$-module $k$.*

*Remark* 4.3. Let $U\mathfrak{g}$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. By definition,

$$\mathscr{F} = U\mathfrak{g} \underset{U\mathfrak{g}_+}{\otimes} k,$$

where $U\mathfrak{g}$ is considered as the right $U\mathfrak{g}_+$-module. Equivalently,

$$\mathscr{F} = \mathscr{W}/\mathscr{D},$$

where $\mathscr{W}$ is the Weyl algebra of $\mathfrak{g}$ — a quotient of $U\mathfrak{g}$ by the ideal generated by $C - \mathbf{1}$, where $\mathbf{1}$ now stands for the unit in $U\mathfrak{g}$, and $\mathscr{D}$ is the left ideal in $\mathscr{W}$ generated by $\mathcal{O}_K \oplus \{0\}$.

Explicit realization of the Fock module $\mathscr{F}$ — "the bosonic Fock space" — depends on the decomposition of $K$ into a direct sum of isotropic subspaces with respect to the bilinear form $c$,

$$K = K_+ \oplus K_-, \tag{4.1}$$

where the subspace $K_+ = \mathcal{O}_K$ is defined canonically. In this case

$$\mathscr{F} \simeq \operatorname{Sym}^\bullet K_-$$

— the symmetric algebra of the $k$-vector space $K_-$. The Fock space $\mathscr{F}$ is a $\mathbb{Z}$-graded commutative algebra

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{F}^{(n)}$$

where $\mathscr{F}^{(n)} \simeq \operatorname{Sym}^n K_-$, $\mathscr{F}^{(0)} = k \cdot \mathbf{1}$, and $\mathscr{F}^{(n)} = \{0\}$ for $n < 0$. For every $f = f_+ + f_- \in K$ the operator $\boldsymbol{f} = \rho(f) \in \operatorname{End} \mathscr{F}$ is defined by

$$\boldsymbol{f} \cdot v = f_- \odot v + \sum_{i=1}^{k} c(f, v_i)\, v^i = f_- \odot v - \sum_{i=1}^{k} \operatorname{Res}\,(f_+\, dv_i)\, v^i, \tag{4.2}$$

where $v = v_1 \odot \cdots \odot v_k \in \mathscr{F}^{(k)}$ and $v^i = v_1 \odot \cdots \odot \hat{v}_i \odot \cdots \odot v_k \in \mathscr{F}^{(k-1)}$, $i = 1, \dots, k$, and $\odot$ denotes for the multiplication in $\operatorname{Sym}^\bullet K_-$, the symmetric tensor product. In particular,

$$\boldsymbol{f} \cdot \mathbf{1} = f_-.$$

The Fock module $\mathscr{F}$ is equipped with the linear topology given by the filtration associated with the $\mathbb{Z}$-grading, which does not depend on the decomposition (4.1).

*Remark* 4.4. Every choice of the uniformizer defines the isomorphism $K \simeq k((t))$, and one can choose $K_- = t^{-1}k[t^{-1}]$. The mapping

$$\mathscr{F}^{(n)} \ni v = t^{-m_1} \odot \cdots \odot t^{-m_n} \mapsto x_{m_1} \dots x_{m_n} \in k[x_1, x_2, \dots]$$

establishes the isomorphism $\mathscr{F} \simeq k[x_1, x_2, \dots]$ between the bosonic Fock space and the polynomial ring in infinitely many variables $\{x_n\}_{n\in\mathbb{N}}$. Under this mapping $\boldsymbol{\alpha}_n \mapsto n\partial/\partial x_n$, $\boldsymbol{\alpha}_{-n} \mapsto x_n$, $n > 0$ — multiplication by $x_n$ operators — and $\boldsymbol{\alpha}_0 \mapsto 0$.

*Remark* 4.5. For a general complete closed field $K$ there is no canonical choice of the isotropic subspace $K_-$ complementary to $K_+ = \mathcal{O}_K$. However, every choice of an effective non-special divisor $D = P_1 + \cdots + P_g$ of degree $g$ on an algebraic curve $X$ and uniformizers $t_i$ at $P_i$, defines the complementary subspaces $K_-$ for all fields $K = F_P$, $P \in X$. Namely, let $\mathcal{A}(X, D)$ be the $k$-vector space of additive functions defined in Example 3.1, and let $\mathcal{A}_P(X, D)$ be the subspace of additive functions with the only pole at $P$. Set

$$K_- = \mathcal{A}_P(X, D)|_P \subset K.$$

According to part (i) of Lemma 3.1, the subspace $K_-$ is isotropic with respect to the bilinear form $c$ and decomposition (4.1) holds. The subspace $K_-$ is spanned by $v_P^{(n)} = \eta_P^{(n)}\Big|_P$, $n \in \mathbb{N}$, and $dK_- = \Omega_0(*P)|_P$.

The bilinear form $c$ has a one-dimensional kernel $k$. Since $\mathcal{O}_K/k = \mathfrak{p}$, the form $c$ defines a non-degenerate continuous pairing $c : \mathfrak{p} \otimes K_- \to k$, so that $\mathfrak{p} = K_-^\vee = \operatorname{Hom}(K_-, k)$ — the topological dual to the $k$-vector space $K_-$. Correspondingly, topological dual to the bosonic Fock space $\mathscr{F}$ is the $k$-vector space $\mathscr{F}^\vee = \overline{\operatorname{Sym}^\bullet \mathfrak{p}}$ — the completion of $\operatorname{Sym}^\bullet \mathfrak{p}$ with respect to the linear topology given by the filtration $\{F^n \operatorname{Sym}^\bullet \mathfrak{p}\}_{n=0}^\infty$,

$$F^n \operatorname{Sym}^\bullet \mathfrak{p} = \oplus_{i=0}^n \operatorname{Sym}^i \mathfrak{p}.$$

The continuous pairing $(\ ,\ ) : \mathscr{F}^\vee \otimes \mathscr{F} \to k$ is uniquely determined by the pairing between $\operatorname{Sym}^\bullet \mathfrak{p}$ and $\mathscr{F} = \operatorname{Sym}^\bullet K_-$, which is defined inductively by

$$(u, v) = \delta_{kl} \sum_{i=1}^{l} c(u_1, v_i)(u^1, v^i), \tag{4.3}$$

where $u = u_1 \odot \cdots \odot u_k = u_1 \odot u^1 \in \operatorname{Sym}^k \mathfrak{p}$, and $v = v_1 \odot \cdots \odot v_l = v_i \odot v^i \in \mathscr{F}^{(l)}$. The dual bosonic Fock space $\mathscr{F}^\vee$ is the right $\mathfrak{g}$-module with the lowest weight vector $\mathbf{1}^\vee$ annihilated by the subspace $K_- \oplus k$.

Explicitly, the representation $\rho$ of $\mathfrak{g}$ in $\mathscr{F}$ defines a contragradient representation $\rho^\vee$ of $\mathfrak{g}$ in $\mathscr{F}^\vee$ by

$$(u \cdot \rho^\vee(f), v) = (u, \rho(f) \cdot v), \text{ for all } u \in \mathscr{F}^\vee,\ v \in \mathscr{F}.$$

Namely, put $f = \tilde{f}_+ + \tilde{f}_- \in K$, where now $\tilde{f}_+ \in \mathfrak{p}$ and $\tilde{f}_- \in K_- \oplus k$. It follows from (4.2) and (4.3) that the operator $\boldsymbol{f} = \rho^\vee(f) \in \operatorname{End} \mathscr{F}^\vee$ is defined by

$$u \cdot \boldsymbol{f} = \tilde{f}_+ \odot u + \sum_{i=1}^{k} c(u_i, f) u^i = \tilde{f}_+ \odot u + \sum_{i=1}^{k} \operatorname{Res}(\tilde{f}_- du_i) u^i, \tag{4.4}$$

where $u = u_1 \odot \cdots \odot u_k \in \operatorname{Sym}^k \mathfrak{p}$ and $u^i = u_1 \odot \cdots \odot \hat{u}_i \odot \cdots \odot u_k \in \operatorname{Sym}^{k-1} \mathfrak{p}$.

4.2. **The lattice algebra.** Let $k[\mathbb{Z}]$ be the group algebra of the additive group $\mathbb{Z}$. As a $k$-vector space, $k[\mathbb{Z}]$ has a basis $\{e_n\}_{n \in \mathbb{Z}}$, $e_m e_n = e_{m+n}$. For every decomposition (4.1), define the "constant term" of $f \in K$ by $f(0) = f_+ \mod \mathfrak{p} \in k$, so that for $f \in K_-$ we have $f(0) = 0$.

*Remark* 4.6. When $K = F_P$ and $K_- = \mathcal{A}_P(X, D)|_P$, $f(0)$ is the constant term of the formal Laurent expansion of $f \in k((t_P))$ with respect to the uniformizer $t_P$ for $K$, defined in Section 3.1.

**Definition.** A lattice algebra $\mathfrak{l}$ associated with the decomposition (4.1) is a semi-direct sum of the Heisenberg Lie algebra $\mathfrak{g}$ and the abelian Lie algebra $k[\mathbb{Z}]$ with the Lie bracket

$$[f + aC + \alpha e_m, g + bC + \beta e_n] = c(f, g)C + \alpha m g(0) e_m - \beta n f(0) e_n,$$

where $f + aC, g + bC \in \mathfrak{g}$.

Corresponding irreducible highest weight module $\mathscr{B}$ for the lattice algebra $\mathfrak{l}$ is given by

$$\mathscr{B} = k[\mathbb{Z}] \odot \mathscr{F},$$

where $k[\mathbb{Z}]$ acts by multiplication, and

$$\boldsymbol{f}(e_n \odot v) = -nf(0)e_n \odot v + e_n \odot \boldsymbol{f} \cdot v, \quad v \in \mathscr{F}.$$

The module $\mathscr{B}$ — the Fock space of "charged bosons" is a $\mathbb{Z}$-graded commutative algebra,

$$\mathscr{B} = \bigoplus_{n\in\mathbb{Z}} \mathscr{B}^{(n)}, \quad \mathscr{B}^{(n)} = k \cdot e_n \odot \mathscr{F}.$$

The elements $e_n$, $n \in \mathbb{Z}$, correspond to the shift operators $\boldsymbol{e}_n = \boldsymbol{e}^n$ in $\mathscr{B}$, where

$$\boldsymbol{e}(e_n \odot v) = e_{n+1} \odot v, \quad v \in \mathscr{F}.$$

*Remark* 4.7. Using canonical isomorphism $K^*/\mathcal{O}_K^* \simeq \mathbb{Z}$ given by the valuation map $v : K^* \to \mathbb{Z}$, the Fock space $\mathscr{B}$ can be also defined as the space of all functions

$$F : K^*/\mathcal{O}_K^* \to \mathscr{F}$$

with finite support.

*Remark* 4.8. For every choice of the uniformizer $t$ for $K$, the mapping

$$\mathscr{B}^{(n)} \ni e_n \odot t^{-m_1} \odot \cdots \odot t^{-m_l} \mapsto e^{nx_0} x_{m_1} \dots x_{m_l} \in e^{nx_0} k[x_1, x_2, \dots]$$

establishes the isomorphism $\mathscr{B} \simeq k[e^{x_0}, e^{-x_0}, x_1, x_2, \dots]$. Under this mapping, $\boldsymbol{\alpha}_n \mapsto n\partial/\partial x_n,\ \boldsymbol{\alpha}_{-n} \mapsto x_n,\ n > 0$, $\boldsymbol{\alpha}_0 \mapsto -\partial/\partial x_0$, and $\boldsymbol{e} \mapsto e^{x_0}$ — a multiplication by the variable $e^{x_0}$ operator.

Topological dual to $\mathscr{B}$ is the $k$-vector space $\mathscr{B}^\vee = \oplus_{n\in\mathbb{Z}}\, k{\cdot}q^n \odot \mathscr{F}^\vee$, where $\{q^n\}_{n\in\mathbb{Z}}$ is the basis in $k[\mathbb{Z}]^\vee$ dual to the basis $\{e_n\}_{n\in\mathbb{Z}}$. The continuous pairing $(\ ,\ ) : \mathscr{B}^\vee \otimes \mathscr{B} \to k$ is given by

$$(q^m \odot u, e_n \odot v) = (u, v)\delta_{mn}, \quad u \in \mathscr{F}^\vee,\ v \in \mathscr{F}.$$

As in the case of the Heisenberg algebra, the representation $\rho$ of $\mathfrak{l}$ in $\mathscr{B}$ defines the contragradient representation $\rho^\vee$ in $\mathscr{B}^\vee$. The dual Fock space $\mathscr{B}^\vee$ is a right $\mathfrak{l}$-module with the lowest weight vector $\mathbf{1}^\vee$ annihilated by $K_-$.

4.3. **The Heisenberg system.** The tame symbol for a complete closed field $K$ is defined by the same formula as in Section 2.4,

$$\tau(f, g) = (-1)^{mn} \frac{f^n}{g^m} \mod \mathfrak{p} \in k^*,$$

where $f, g \in K^*$ and $m = v(f), n = v(g)$. Let $G$ be the central extension of the multiplicative group $K^*$ by the tame symbol, $G \simeq K^* \times k^*$ with the group law

$$(f_1, \alpha_1)(f_2, \alpha_2) = (f_1 f_2, \tau(f_1, f_2)^{-1}\alpha_1\alpha_2),$$

where $f_1, f_2 \in K^*$ and $\alpha_1, \alpha_2 \in k^*$. The group $G$ is a topological group with the topology defined by the decomposition $G \simeq \mathbb{Z} \times \mathcal{O}_K^* \times k^*$, where $k^*$ and $\mathbb{Z}$ have discrete topology, and $\mathcal{O}_K^*$ has $\mathfrak{p}$-adic topology.

**Definition.** A $G$-module is a $k$-vector space $V$, with the discrete topology and with a group homomorphism $R : G \to \operatorname{End} V$, such that $G$-action on $V$ is continuous and $R((1, \alpha)) = \alpha \boldsymbol{I}$, $\alpha \in k^*$.

For $f \in K^*$ setting $R(f) = R((f, 1)) \in \operatorname{End} V$, we get

$$R(f_1)R(f_2) = \tau(f_1, f_2)R(f_1 f_2)$$

— a projective representation of the multiplicative group $K^*$. Continuity means that for every $v \in V$ there exists a open subspace $U$ in $K$, commensurable with $\mathfrak{p}$, such that $U^* \times \{1\}$ fixes $v$, $R(U^*)v = v$.

Though the Heisenberg algebra $\mathfrak{g}$ is not a Lie algebra of the group $G$, there is an "adjoint action" of $G$ on $\mathfrak{g}$, defined by

$$\operatorname{Ad} g \cdot x = x + \operatorname{Res}(f d \log h)\, C,$$

where $g = (h, \alpha) \in G$, $x = f + a\, C \in \mathfrak{g}$. Following Garland and Zuckerman [GZ91], we call the triple $(\mathfrak{g}, G, \operatorname{Ad})$ a Heisenberg system.

Denote by $G_+ = \mathcal{O}_K^* \times k^*$ the maximal abelian subgroup of $G$. Since the field $k$ has characteristic 0, we have $\mathcal{O}_K^* \simeq k^* \times \exp \mathfrak{p}$. The representation $\rho$ of $\mathfrak{g}$ in $\mathscr{F}$, constructed in Section 4.1, defines a representation $r$ of $G_+$ in $\mathscr{F}$ by the formula

$$r(g_+) = \beta \exp \rho(\varphi) = \beta \exp \boldsymbol{\varphi}, \quad g_+ = (\alpha \exp \varphi, \beta) \in G_+,$$

where $\varphi \in \mathfrak{p}$ and $\alpha, \beta \in k^*$. Since $\mathscr{F}$ is the highest weight module with respect to the abelian subalgebra $\mathcal{O}_K + \{0\}$ of $\mathfrak{g}$, the operators $r(g_+) \in \operatorname{End} \mathscr{F}$ are well-defined.

**Definition.** A representation of the Heisenberg system $(G, \mathfrak{g}, \operatorname{Ad})$ in a vector space $V$ is the pair $(R, dR)$, where $R : G \to \operatorname{End} V$ is a representation of the group $G$, and $dR : \mathfrak{g} \to \operatorname{End} V$ is a representation of the Heisenberg Lie algebra $\mathfrak{g}$, satisfying $dR(\operatorname{Ad} g \cdot x) = R(g) dR(x) R(g)^{-1}$.

Let $R$ be a representation of $G$ induced by the representation $r$ of the subgroup $G_+$. Explicitly, the $G$-module $\operatorname{Ind}_{G_+}^G \mathscr{F}$ consists of all functions $F : G \to \mathscr{F}$, satisfying

$$F(g g_+) = r(g_+)^{-1} F(g), \quad g_+ \in G_+, \, g \in G,$$

and such that corresponding sections over $G/G_+ \simeq \mathbb{Z}$ have finite support. The representation $R$ is given by

$$R(g)F(h) = F(g^{-1}h), \quad g, h \in G.$$

Define the representation $dR$ of the Heisenberg Lie algebra $\mathfrak{g}$ by

$$(dR(x) \cdot F)(g) = \rho(\operatorname{Ad}\, g^{-1} \cdot x)(F(g)), \quad x \in \mathfrak{g}, \, g \in G.$$

**Theorem 4.2** (Garland-Zuckerman)**.** *The pair* $(R, dR)$ *is a representation of the Heisenberg system* $(\mathfrak{g}, G, \mathrm{Ad})$*, and*

$$\mathrm{Ind}_{G_+}^{G} \mathscr{F} \simeq \mathscr{B}.$$

*Proof.* See [GZ91, Sec. 3]. The isomorphism $\mathrm{Ind}_{G_+}^{G} \mathscr{F} \simeq \mathscr{B}$ follows from the first remark in Section 4.2. □

*Remark* 4.9. Every choice of the uniformizer for $K$ defines the group isomorphism $K^* \simeq k((t))^* = k^* \times \mathbb{Z} \times tk[[t]]$. Explicitly, every $f \in K^*$ can be uniquely written in the form

$$f = \alpha t^n \exp \varphi, \quad \alpha \in k^*, \ \varphi \in tk[[t]].$$

In particular, when $K = F_P$, $P \in X$, there is a natural choice of the uniformizer $t = t_P$ associated with the choice of an effective non-special divisor $D = P_1 + \cdots + P_g$ of degree $g$ on $X$, uniformizers $t_i$ at $P_i$, and additive functions $\eta_i$ (see Section 3.1). Identifying the elements $e_m \odot v \in \mathscr{B}$ with the functions $F : \mathbb{Z} \to \mathscr{F}$ defined by $F(n) = \delta_{mn} v$, $n \in \mathbb{Z}$, we obtain by a straightforward computation (see [GZ91, Sec. 4]) that for $f = \alpha t^n \exp \varphi \in K^*$,

$$R(f)(e_m \odot v) = (-1)^{mn} \alpha^{-n-2m} e_{m+n} \odot \exp \boldsymbol{\varphi} \cdot v.$$

Similarly,

$$dR(f)(e_m \odot v) = -m f(0) e_m \odot v + e_m \odot \boldsymbol{f} \cdot v,$$

where now $f \in K$.

As in the case of the lattice algebra, the representation $(R, dR)$ in $\mathscr{B}$ defines the contragradient representation $(R^\vee, dR^\vee)$ in the dual Fock space $\mathscr{B}^\vee$. In particular,

$$(q^m \odot u) R^\vee(f) = (-1)^{(m-n)n} \alpha^{n-2m} q^{m-n} \odot u \cdot \exp \boldsymbol{\varphi}, \quad f = \alpha t^n \exp \varphi \in K^*.$$

## 5. Global Theory

Here for an algebraic curve $X$ over algebraically closed field $k$ of characteristic zero we define global versions of local QFT's introduced in the previous section. Succinctly, these global QFT's can be characterized as follows.

**1.** "QFT of additive bosons on $X$", which corresponds to the global Heisenberg algebra $\mathfrak{g}_X$ — the restricted direct sum of local Heisenberg algebras $\mathfrak{g}_P$ over all points $P \in X$. The global Fock space $\mathscr{F}_X$ is defined as the restricted symmetric tensor product of local Fock spaces $\mathscr{F}_P$ over all points $P \in X$. The global Fock space $\mathscr{F}_X$ is the highest weight $\mathfrak{g}_X$-module, and there exists a linear functional $\langle \cdot \rangle : \mathscr{F}_X \to k$ — "the expectation value" functional, uniquely characterized by its normalization and the invariance property with respect to the space of additive functions.

**2.** "QFT of charged bosons on $X$", which corresponds to global lattice algebra $\mathfrak{l}_X$. The global charged Fock space $\mathscr{B}_X$ is the highest weight $\mathfrak{l}_X$-module, and there exists a unique expectation value functional $\langle\cdot\rangle : \mathscr{B}_X \to k$ with similar properties.

**3.** "QFT of multiplicative bosons on $X$", which corresponds to the action of the global Heisenberg system $(G_X, \mathfrak{g}_X, \mathrm{Ad})$ on $\mathscr{B}_X$ with the property that the expectation value functional is invariant under the group of multiplicative functions. The latter is equivalent to the generalized A. Weil reciprocity law on algebraic curves.

5.1. **Additive bosons on $X$.** The theory consists of the following data.

**AB1** Non-special effective divisor $D_{\mathrm{ns}} = P_1 + \cdots + P_g$ of degree $g$ on $X$ with distinct points, uniformizers $t_i$ at $P_i$, and the $k$-vector space of additive functions $\mathcal{A}(X, D_{\mathrm{ns}})$ — a subspace of $\mathbb{A}_X$ containing $F = k(X)$, introduced in Example 3.1.

**AB2** Local QFT's of additive bosons — highest weight $\mathfrak{g}_P$-modules $\mathscr{F}_P$ for all points $P \in X$.

**AB3** Global Heisenberg algebra $\mathfrak{g}_X$ — a one-dimensional central extension of the abelian Lie algebra $\mathfrak{gl}_1(\mathbb{A}_X) = \mathbb{A}_X$ by the cocycle $c_X = \sum_{P\in X} c_P$.

**AB4** The highest weight $\mathfrak{g}_X$-module — the global Fock space $\mathscr{F}_X$ — a restricted symmetric tensor product of $\mathscr{F}_P$ over all points $P \in X$.

**AB5** The expectation value functional — the linear mapping $\langle\cdot\rangle : \mathscr{F}_X \to k$, satisfying the following properties:
- (i) $\langle \mathbf{1}_X\rangle = 1$, where $\mathbf{1}_X \in \mathscr{F}_X$ is the highest weight vector.
- (ii) $\langle \boldsymbol{a}\cdot v\rangle = 0$ for all $a \in \mathcal{A}(X, D_{\mathrm{ns}})$ and $v \in \mathscr{F}_X$.

Parts **AB1** and **AB2** of the theory have been described in Sections 3.1 and 4.1. Here we introduce the global Heisenberg algebra $\mathfrak{g}_X$, construct the corresponding global Fock space $\mathscr{F}_X$, and prove that the expectation value functional $\langle\cdot\rangle$ satisfying properties (i) and (ii) exists and is unique.

Let $c_X : \mathbb{A}_X \times \mathbb{A}_X \to k$ be the global bilinear form,

$$c_X(x, y) = \sum_{P\in X} c_P(x_P, y_P) = -\sum_{P\in X} \mathrm{Res}_P(x_P dy_P), \quad x, y \in \mathbb{A}_X.$$

**Definition.** The global Heisenberg Lie algebra $\mathfrak{g}_X$ is a one-dimensional central extension of the abelian Lie algebra $\mathbb{A}_X$

$$0 \to k\,C \to \mathfrak{g}_X \to \mathbb{A}_X \to 0$$

by the two-cocycle $c_X$.

The Lie subalgebra $\mathfrak{g}_X^+ = \mathcal{O}_X \oplus kC$, where $\mathcal{O}_X = \prod_{P\in X} \mathcal{O}_P$, is the maximal abelian subalgebra of $\mathfrak{g}_X$.

**Definition.** The global Fock space $\mathscr{F}_X$ is the irreducible $\mathfrak{g}_X$-module with the vector $\mathbf{1}_X$ annihilated by the abelian subalgebra $\mathcal{O}_X \oplus \{0\}$.

As in the local case, the global Fock module is induced from the one-dimensional $\mathfrak{g}_X^+$–module,

$$\mathscr{F}_X = \operatorname{Ind}_{\mathfrak{g}_X^+}^{\mathfrak{g}_X} k.$$

According to the previous section, for $K = F_P$, $P \in X$, we have a decomposition (4.1), where $F_P^{(+)} = \mathcal{O}_P$ and $F_P^{(-)} = \mathcal{A}_P(X, D)|_P$. This gives the following decomposition of the $k$-vector space $\mathbb{A}_X$ into the direct sum of the isotropic subspaces with respect to the bilinear form $c_X$,

$$\mathbb{A}_X = \mathcal{O}_X \oplus \mathcal{F}_X^{(-)}, \tag{5.1}$$

where

$$\mathcal{F}_X^{(-)} = \coprod_{P\in X} F_P^{(-)}$$

— a restricted direct product over all $P \in X$ with respect to the zero subspaces $\{0\} \subset F_P^{(-)}$. The decomposition (5.1) gives rise to the isomorphism

$$\mathscr{F}_X \simeq \operatorname{Sym}^\bullet \mathcal{F}_X^{(-)}.$$

The global Fock space $\mathscr{F}_X$ carries a linear topology given by the natural filtration associated with the $\mathbb{Z}$-grading.

Equivalently, $\mathscr{F}_X$ can be defined as the symmetric tensor product

$$\mathscr{F}_X = \widehat{\bigodot_{P\in X}} \mathscr{F}_P,$$

restricted with respect to the vectors $\mathbf{1}_P \in \mathscr{F}_P$, equipped with the product topology. In other words, $\mathbf{1}_X = \odot_{P\in X}\mathbf{1}_P$, and $\mathscr{F}_X$ is spanned by the vectors

$$v = \bigodot_{P\in X} v_P,$$

where $v_P = \mathbf{1}_P$ for all but finitely many $P \in X$. For every $P \in X$ we have $v = v_P \odot v^P$, where $v^P = \odot_{Q\in X}\tilde{v}_Q$, $\tilde{v}_Q = v_Q$ for $Q \neq P$ and $\tilde{v}_P = \mathbf{1}_P$. Denote by $\rho_P$ corresponding representation of $\mathfrak{g}_P$ in $\mathscr{F}_P$, $P \in X$, and by $\rho$ — the representation of $\mathfrak{g}_X$ in $\mathscr{F}_X$. Setting $\boldsymbol{x} = \rho(x) \in \operatorname{End}\mathscr{F}_X$ for $x = \{x_P\}_{P\in X} \in \mathbb{A}_X$, we have for $v = \odot_{P\in X} v_P$,

$$\boldsymbol{x} \cdot v = \sum_{P\in X} \boldsymbol{x}_P \cdot v_P \odot v^P,$$

where $\boldsymbol{x}_P = \rho_P(x_P) \in \operatorname{End}\mathscr{F}_P$.

Set

$$\mathfrak{P}_X = \prod_{P\in X} \mathfrak{p}.$$

Topological dual to the global Fock space $\mathscr{F}_X$ is the $k$-vector space $\mathscr{F}_X^\vee = \overline{\operatorname{Sym}^\bullet \mathfrak{P}_X}$ — the completion of $\operatorname{Sym}^\bullet \mathfrak{P}_X$ with respect to the linear topology given by the natural filtration associated with the $\mathbb{Z}$-grading. The dual global Fock space $\mathscr{F}_X^\vee$ is the right $\mathfrak{g}_X$-module with a lowest weight vector $\mathbf{1}_X^\vee$ annihilated by the abelian subalgebra $\mathcal{F}_X^{(-)} \oplus \{0\}$. Equivalently,

$$\mathscr{F}_X^\vee = \overline{\widehat{\bigodot_{P\in X}} \mathscr{F}_P^\vee}$$

— the completion of the symmetric tensor product restricted with respect to the vectors $\mathbf{1}_P^\vee$. The completion is taken with respect to the double filtration $\{F^{mn}\operatorname{Sym}^\bullet \mathfrak{P}_X\}$,

$$F^{mn}\operatorname{Sym}^\bullet \mathfrak{P}_X = \sum_{i=0}^{m}\sum_{P_1,\dots,P_i\in X}\left(\bigoplus_{l_1+\dots+l_i=0}^{n}\operatorname{Sym}^{l_1}\mathfrak{p}_1\odot\dots\odot\operatorname{Sym}^{l_i}\mathfrak{p}_i\right).$$

In other words, the elements of $\mathscr{F}_X^\vee$ are infinite sums

$$u=\sum_{n=0}^{\infty}\sum_{P_1,\dots,P_n\in X}a_{P_1\dots P_n}u_{P_1\dots P_n},$$

where $u_{P_1\dots P_n}\in\overline{\mathscr{F}^\vee}_{P_1\dots P_n}$ — a completion of the symmetric tensor product

$$\mathscr{F}^\vee_{P_1\dots P_n}=\mathscr{F}^\vee_{P_1}\odot\dots\odot\mathscr{F}^\vee_{P_n}$$

with respect to the filtration

$$F^m\mathscr{F}^\vee_{P_1\dots P_n}=\bigoplus_{l_1+\dots+l_n=0}^{m}\left(\operatorname{Sym}^{l_1}\mathfrak{p}_1\odot\dots\odot\operatorname{Sym}^{l_n}\mathfrak{p}_n\right).$$

Denote by $\{u_P^{(n)}\}_{n\in\mathbb{N}}$ the basis for $\mathfrak{p}$ dual to the basis $\left\{v_P^{(n)}=\eta_P^{(n)}\Big|_P\right\}_{n\in\mathbb{N}}$ for $F_P^{(-)}$ with respect to the pairing given by $c_P$ (see Section 4.1). Then we obtain that $\mathscr{F}_X^\vee$ is a completion of $k[[u_P^n]]$ — the ring of formal Taylor series in infinitely many variables $u_P^{(n)}$, $P\in X, n\in\mathbb{N}$. This realization of $\mathscr{F}_X^\vee$ is used to prove the following main result for the QFT of additive bosons.

**Theorem 5.1.** *There exists a unique linear functional $\langle\,\cdot\,\rangle:\mathscr{F}_X\to k$ — the expectation value functional — satisfying the following properties.*

**EV1** $\langle\mathbf{1}_X\rangle=1$.

**EV2** $\langle\boldsymbol{a}\cdot v\rangle=0$ *for all* $a\in\mathcal{A}(X,D_{\mathrm{ns}})$ *and* $v\in\mathscr{F}_X$.

*The functional $\langle\,\cdot\,\rangle$ has the form*

$$\langle v\rangle=(\Omega_X,v),$$

*where*

$$\Omega_X=\exp\left\{-\frac{1}{2}\sum_{m,n=1}^{\infty}\sum_{P,Q\in X}c_{PQ}^{(mn)}u_P^{(m)}u_Q^{(n)}\right\}\in\mathscr{F}_X^\vee,$$

*and*

$$c_{PQ}^{(mn)}=-\operatorname{Res}_Q(\eta_P^{(m)}d\eta_Q^{(n)}).$$

*Proof.* It follows from decomposition (3.5) that the linear functional $\langle v\rangle=(\Omega,v)$ verifies properties **EV1** and **EV2** if and only if it is normalized, $(\Omega,\mathbf{1}_X)=1$, and $\Omega\in\mathscr{F}_X^\vee$ satisfies the equations

$$\Omega\cdot\boldsymbol{\eta}_{\boldsymbol{P}}^{(\boldsymbol{n})}=0\tag{5.2}$$

for all $P\in X$ and $n\in\mathbb{N}$, where $\boldsymbol{\eta_P^{(n)}}=\rho^\vee(\eta_P^{(n)})$. Let $\eta_P^{(n)}=\beta_P^{(n)}+\gamma_P^{(n)}$, where $\beta_P^{(n)}=\{\beta_{PQ}^{(n)}\}_{Q\in X},\gamma_P^{(n)}=\{\gamma_{PQ}^{(n)}\}_{Q\in X}\in\mathbb{A}_X$ are given by

$$\beta_{PQ}^{(n)}=\begin{cases}0 & \text{if } Q=P,\\ \eta_P^{(n)}\Big|_Q & \text{if } Q\neq P,\end{cases}$$

and

$$\gamma_{PQ}^{(n)}=\begin{cases}\eta_P^{(n)}\Big|_P & \text{if } Q=P,\\ 0 & \text{if } Q\neq P.\end{cases}$$

It follows from (4.4) that $\boldsymbol{\gamma_P^{(n)}}$ acts on $\mathscr{F}_X^\vee$ as a differentiation with respect to the variable $u_P^{(n)}$. For $Q\neq P$ we have

$$\beta_{PQ}^{(n)}=a_{PQ}^{(n)}+\sum_{m=1}^{\infty}a_{PQ}^{(nm)}u_Q^{(m)},$$

where $a_{PQ}^{(n)}\in k$ and

$$a_{PQ}^{(nm)}=c(\beta_{PQ}^{(n)},v_Q^{(m)})=-\operatorname{Res}_Q(\eta_P^{(n)}d\eta_Q^{(m)})=c_{PQ}^{(nm)}.$$

Since $c_{PP}^{(nm)}=0$ (see Lemma 3.1) we conclude that $\boldsymbol{\beta_P^{(n)}}$ acts on $\mathscr{F}_X^\vee$ as a multiplication by $\sum_{Q\in X}c_{PQ}^{(nm)}u_Q^{(m)}$. The equations (5.2) can be rewritten as

$$\left(\frac{\partial}{\partial u_P^{(n)}}+\sum_{Q\in X}c_{PQ}^{(nm)}u_Q^{(m)}\right)\Omega=0,\quad P\in X,\ n\in\mathbb{N}. \tag{5.3}$$

As it follows from part (i) of Lemma 3.1,

$$c_{PQ}^{(mn)}=c_{QP}^{(nm)},$$

so that the system of differential equations (5.3) is compatible and $\Omega_X$ is its unique normalized solution. □

5.2. **Charged additive bosons on $X$.** The theory consists of the following data.

- **CB1** Non-special effective divisor $D_{\rm ns}=P_1+\cdots+P_g$ of degree $g$ on $X$ with distinct points, uniformizers $t_i$ at $P_i$, and the $k$-vector space of additive functions $\mathcal{A}(X,D_{\rm ns})$ , a subspace of $\mathbb{A}_X$ containing $F=k(X)$, introduced in Example 3.1.
- **CB2** Local QFT's of charged additive bosons — highest weight $\mathfrak{l}_P$-modules $\mathscr{B}_P$ for all points $P\in X$.
- **CB3** Global lattice algebra $\mathfrak{l}_X$ — the semi-direct sum of the global Heisenberg algebra $\mathfrak{g}_X$ and the abelian Lie algebra $k[\mathrm{Div}_0(X)]$ with generators $e_D$, $D\in\mathrm{Div}_0(X)$ — the group algebra of the additive group $\mathrm{Div}_0(X)$ of degree 0 divisors on $X$.

**CB4** The highest weight $\mathfrak{l}_X$-module — the global Fock space $\mathscr{B}_X$ with the highest weight vector $\mathbf{1}_X \in \mathscr{B}_X$.

**CB5** The expectation value functional — the linear mapping $\langle \cdot \rangle : \mathscr{B}_X \to k$, satisfying the following properties:
  (i) $\langle \boldsymbol{e_D} \cdot \mathbf{1}_X \rangle = 1$ for all $D \in \mathrm{Div}_0(X)$.
  (ii) $\langle \boldsymbol{a} \cdot u \rangle = 0$ for all $a \in \mathcal{A}(X, D_{\mathrm{ns}})$ and $u \in \mathscr{B}_X$.

As a $k$-vector space, the group algebra $k[\mathrm{Div}_0(X)]$ of the additive group $\mathrm{Div}_0(X)$ of degree 0 divisors on $X$ has a basis $\{e_D\}_{D \in \mathrm{Div}_0(X)}$, $e_{D_1} e_{D_2} = e_{D_1+D_2}$. For every $x = \{x_P\} \in \mathbb{A}_X$ and $D = \sum_{P \in X} n_P \, P \in \mathrm{Div}_0(X)$ we put

$$x(D) = \sum_{P \in X} n_P x_P(0) \in k,$$

where $x_P(0) = x_P^+ \bmod \mathfrak{p} \in k$ is the constant term of $x_P \in F_P$, defined by the decomposition (4.1) associated with the non-special divisor $D_{\mathrm{ns}}$ (see Section 4.2).

**Definition.** The global lattice algebra $\mathfrak{l}_X$ is a semi-direct sum of the global Heisenberg algebra $\mathfrak{g}_X$ and the abelian Lie algebra $k[\mathrm{Div}_0(X)]$ with the Lie bracket

$$[x + \alpha C + \gamma e_{D_1}, y + \beta C + \delta e_{D_2}] = c_X(x, y) C + y(D_1)\gamma e_{D_1} - x(D_2)\delta e_{D_2},$$

where $x + \alpha C, y + \beta C \in \mathfrak{g}_X$, $\gamma, \delta \in k$.

The global Fock space $\mathscr{B}_X$ is a symmetric tensor product of the group algebra $k[\mathrm{Div}_0(X)]$ and the Fock space of additive bosons $\mathscr{F}_X$,

$$\mathscr{B}_X = k[\mathrm{Div}_0(X)] \odot \mathscr{F}_X = \bigoplus_{D \in \mathrm{Div}_0(X)} \mathscr{B}_X^D,$$

where

$$\mathscr{B}_X^D = k \cdot e_D \odot \mathscr{F}_X.$$

The global Fock space $\mathscr{B}_X$ is the irreducible $\mathfrak{l}_X$-module, where $k[\mathrm{Div}_0(X)]$ acts by multiplication,

$$\boldsymbol{e_{D_1}}(e_{D_2} \odot v) = e_{D_1+D_2} \odot v, \quad v \in \mathscr{F}_X \tag{5.4}$$

and

$$\boldsymbol{x}(e_D \odot v) = -x(D) e_D \odot v + e_D \odot \boldsymbol{x} \cdot v, \quad v \in \mathscr{F}_X. \tag{5.5}$$

For every $D = \sum_{P \in X} n_P \, P \in \mathrm{Div}_0(X)$ the subspace $\mathscr{B}_X^D$ is the irreducible $\mathfrak{g}_X$-module. It has the property that for $x = \{x_P\}_{P \in X} \in \mathbb{A}_X$ such that $x_P \in k$ for all $P \in X$, the restriction of the operator $\boldsymbol{x}$ to the subspace $\mathscr{B}_X^D$ is equal to $-x(D)\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity operator. In particular, when $x = c$ is a constant, $x(D) = c \deg D = 0$, and $\boldsymbol{x}$ acts by zero in $\mathscr{B}_X$.

*Remark* 5.1. One can also define the extended global lattice algebra $\tilde{l}_X$ as a semi-direct sum of the global Heisenberg algebra $\mathfrak{g}_X$ and the abelian Lie

algebra $k[\mathrm{Div}(X)]$, as well as its irreducible module — the extended Fock space

$$\tilde{\mathscr{B}}_X = k[\mathrm{Div}(X)] \odot \mathscr{F}_X = \bigoplus_{D\in\mathrm{Div}(X)} \mathscr{B}_X^D.$$

The action of $\tilde{l}_X$ in $\tilde{\mathscr{B}}_X$ is given by as the same formulas (5.4)–(5.5), where now the constant adele $x = c$ acts in $\mathscr{B}_X^D$ by $(c \deg D)\boldsymbol{I}$.

The dual Fock space $\mathscr{B}_X^\vee$ is defined as a completion of the direct sum of the dual spaces to $\mathscr{B}_X^D$ over $D \in \mathrm{Div}_0(X)$, given by the formal infinite sums. Explicitly,

$$\mathscr{B}_X^\vee = \overline{\bigoplus_{D\in\mathrm{Div}_0(X)} \mathscr{B}_X^\vee(D)},$$

where

$$\mathscr{B}_X^\vee(D) = k \cdot q^D \odot \mathscr{F}_X^\vee,$$

$q^D \in k[\mathrm{Div}_0(X)]^\vee$ are dual to $e_D$, and $\mathscr{F}_X^\vee$ was defined in Section 5.1.

**Theorem 5.2.** *There exists a unique linear functional $\langle\,\cdot\,\rangle : \mathscr{B}_X \to k$ — the expectation value functional — satisfying the following properties:*

**EV1** $\langle \boldsymbol{e_D} \cdot \mathbf{1}_X\rangle = 1$ *for all* $D \in \mathrm{Div}_0(X)$.
**EV2** $\langle \boldsymbol{a} \cdot v\rangle = 0$ *for all* $a \in \mathcal{A}(X, D_{\mathrm{ns}})$ *and* $v \in \mathscr{B}_X$.

*The functional $\langle\,\cdot\,\rangle$ has the form*

$$\langle v\rangle = (\hat{\Omega}_X, v),$$

*where*

$$\hat{\Omega}_X = \sum_{D\in\mathrm{Div}_0(X)} q^D \odot \exp\left\{\sum_{n=1}^{\infty}\sum_{P\in X} \eta_P^{(n)}(D)u_P^{(n)}\right\}\Omega_X \in \mathscr{B}_X^\vee,$$

*and $\Omega_X$ is given in Theorem 5.1.*

*Proof.* As in the proof of Theorem 5.1, put

$$\Omega = \sum_{D\in\mathrm{Div}_0(X)} q^D \odot \Omega_D, \quad \Omega_D \in \mathscr{F}_X^\vee.$$

Condition $(\Omega, e_D \odot \mathbf{1}_X) = 1$ for all $D \in \mathrm{Div}_0(X)$ is equivalent to the normalization $(\Omega_D, \mathbf{1}_X) = 1$. The constants act by zero in $\mathscr{B}_X$, so it is sufficient to verify the equations

$$(q^D \odot \Omega_D) \cdot \boldsymbol{\eta_P^{(n)}} = 0 \tag{5.6}$$

for all $D = \sum_{Q\in X} n_Q\, Q \in \mathrm{Div}_0(X)$ and $P \in X$. Since

$$q^D \cdot \boldsymbol{\eta_P^{(n)}} = -\eta_P^{(n)}(D)\, q^D = -\sum_{Q\in X} n_Q\, \eta_P^{(n)}\Big|_Q (0)\, q^D$$

(note that, by definition in Section 4.2, $\eta_P^{(n)}\Big|_P(0)=0$), we get from (5.6) that $\Omega_D$ satisfies the following system of differential equations

$$\left(\frac{\partial}{\partial u_P^{(n)}}-\sum_{Q\in X}n_Q\,\eta_P^{(n)}\Big|_Q(0)+\sum_{Q\in X}c_{PQ}^{(nm)}u_Q^{(m)}\right)\Omega_D=0,$$

which has a unique normalized solution given by

$$\Omega_D=\exp\left\{\sum_{n=1}^{\infty}\sum_{P\in X}\eta_P^{(n)}(D)u_P^{(n)}-\frac{1}{2}\sum_{m,n=1}^{\infty}\sum_{P,Q\in X}c_{PQ}^{(mn)}u_P^{(m)}u_Q^{(n)}\right\}. \qquad \square$$

5.3. **Multiplicative bosons on $X$.** The theory consists of the following data.

**MB1** Non-special effective divisor $D_{\rm ns}=P_1+\cdots+P_g$ of degree $g$ on $X$ with distinct points, uniformizers $t_i$ at $P_i$, and the group of multiplicative functions $\mathcal{M}(X,D)\subset\mathbb{J}_X$ associated with the $k$-vector space of additive functions $\mathcal{A}(X,D_{\rm ns})$, introduced in Example 3.2.

**MB2** Local QFT's of multiplicative bosons — representations $(R_P,dR_P)$ in $\mathscr{B}_P$ of the local Heisenberg systems $(G_P,\mathfrak{g}_P,\mathrm{Ad})$ for all points $P\in X$.

**MB3** The global group $G_X^0$ — a one-dimensional central extension of the subgroup $\mathbb{J}_X^0$ of degree zero ideles by the global tame symbol $\tau_X$ — and the global Heisenberg system $(G_X^0,\mathfrak{g}_X,\mathrm{Ad})$.

**MB4** The representation $(R_X,dR_X)$ of the global Heisenberg system in the global Fock space $\mathscr{B}_X$.

**MB5** The expectation value functional — the linear mapping $\langle\cdot\rangle:\mathscr{B}_X\to k$, satisfying the following properties:
- (i) $\langle\mathbf{1}_X\rangle=1$.
- (ii) $\langle\boldsymbol{x}\cdot v\rangle=0$ for all $x\in\mathcal{A}(X,D_{\rm ns})$ and $v\in\mathscr{B}_X$.
- (iii) $\langle\boldsymbol{m}\cdot v\rangle=\langle v\rangle$ for all $m\in\mathcal{M}(X,D_{\rm ns})$ and $v\in\mathscr{B}_X$.

Parts **MB1** and **MB2** were described in Sections 3.1 and 4.3.

**Definition.** The global group $G_X$ is a central extension of the group of ideles $\mathbb{J}_X$ by the global tame symbol $\tau_X$ — $G_X\simeq\mathbb{J}_X\times k^*$ — with the group law

$$(a,\alpha)(b,\beta)=(ab,\tau_X(a,b)^{-1}\alpha\beta),\quad a,b\in\mathbb{J}_X,\ \alpha,\beta\in k^*,$$

where

$$\tau_X(a,b)=\prod_{P\in X}\tau_P(a_P,b_P).$$

The "adjoint action" of the global group $G_X$ on the global Heisenberg algebra $\mathfrak{g}_X$ is defined by

$$\mathrm{Ad}\,g\cdot\tilde{x}=\tilde{x}+\sum_{P\in X}\mathrm{Res}_P(x_Pd\log a_P)C,$$

where $g = (\{a_P\}_{P\in X}, \alpha) \in G_X$, $\tilde{x} = x + \gamma C \in \mathfrak{g}_X$ and $x = \{x_P\}_{P\in X} \in \mathbb{A}_X$. The triple $(G_X, \mathfrak{g}_X, \mathrm{Ad})$ is called the global Heisenberg system. By definition, representation $(R_X, dR_X)$ of the global Heisenberg system $(G_X, \mathfrak{g}_X, \mathrm{Ad})$ is a pair $(R_X, dR_X)$, where $R$ is a representation of the group $G_X$, and $dR_X$ is a representation the Lie algebra $\mathfrak{g}_X$ satisfying $dR_X(\mathrm{Ad}\, g \cdot \tilde{x}) = R(g)dR_X(\tilde{x})R(g)$.

As in the local case (see Section 4.3), representation $(R_X, dR_X)$ of the global Heisenberg system $(G_X, \mathfrak{g}_X, \mathrm{Ad})$ is induced by representation of the abelian subgroup $G_X^+ = \prod_{P\in X} \mathcal{O}_P^* \times k^*$. Namely, every $g_+ \in G_X^+$ can be written as

$$g_+ = (\{\alpha_P \exp \varphi_P\}_{P\in X}, \beta),$$

where $\alpha_P, \beta \in k^*$, $\varphi_P \in \mathfrak{p}$ for all $P \in X$, and we define a representation $r_X$ of $G_X^+$ in $\mathscr{F}_X$ by

$$r_X(g_+) = \beta \exp \rho(\varphi) = \beta \exp \boldsymbol{\varphi}, \quad \boldsymbol{\varphi} = \{\boldsymbol{\varphi_P}\}_{P\in X} \in \mathrm{End}\, \mathscr{F}_X.$$

Explicitly,

$$\exp \boldsymbol{\varphi} \cdot v = \mathop{\odot}_{P\in X} \exp \boldsymbol{\varphi_P} \cdot v_P, \quad v = \mathop{\odot}_{P\in X} v_P \in \mathscr{F}_X.$$

As in Section 4.3, the operators $r(g_+) \in \mathrm{End}\, \mathscr{F}_X$ are well-defined since $\varphi_P \in \mathfrak{p}$ for all $P \in X$. The $G_X$-module $\mathrm{Ind}_{G_X^+}^{G_X} \mathscr{F}_X$ consists of all functions $F : G_X \to \mathscr{F}_X$ satisfying

$$F(gg_+) = r_X(g_+)^{-1} F(g), \quad g_+ \in G_X^+,\ g \in G_X$$

and such that corresponding sections over $G_X/G_X^+ \simeq \mathrm{Div}(X)$ have finite support. The representation $R_X$ is given by

$$R_X(g)F(h) = F(g^{-1}h), \quad g, h \in G_X,$$

and the corresponding representation $dR_X$ of the global Heisenberg algebra $\mathfrak{g}_X$ is given by

$$(dR_X(\tilde{x})F)(g) = \rho(\mathrm{Ad}\, g^{-1} \cdot \tilde{x})(F(g)), \quad \tilde{x} \in \mathfrak{g}_X,\ g \in G_X.$$

We summarize these results as the global analog of Theorem 4.2.

**Theorem 5.3.** *The pair $(R_X, dR_X)$ is a representation of the global Heisenberg system $(G_X, \mathfrak{g}_X, \mathrm{Ad})$, and*

$$\mathrm{Ind}_{G_X^+}^{G_X} \mathscr{F}_X \simeq \tilde{\mathscr{B}}_X.$$

Explicit construction of the representations $R_X$ and $dR_X$ is the following. We identify the elements $e_D \odot v \in \tilde{\mathscr{B}}_X$ with the functions $F : \mathrm{Div}(X) \to \mathscr{F}_X$ defined by

$$F(D') = \begin{cases} v, & \text{if } D' = D, \\ 0, & \text{otherwise} \end{cases}$$

and using the uniformizers $t_P$, introduced in Section 3.1, represent every idele $a=\{a_P\}_{P\in X}\in\mathbb{J}_X$ as

$$a=\left\{\alpha_P t_P^{v_P(a_P)}\exp\varphi_P\right\}_{P\in X},\quad\text{where}\quad \alpha_P\in k^*,\ \varphi_P\in\mathfrak{p}.$$

Then the representation $R_X$ is defined by

$$\begin{aligned}\boldsymbol{a}\cdot(e_D\odot v)&=R_X((a,1))(e_D\odot v)\\ &=(-1)^{\sum_{P\in X}v_P(D_a)v_P(D)}\prod_{P\in X}\alpha_P^{-v_P(D_a)-2v_P(D)}e_{D+D_a}\odot\exp\boldsymbol{\varphi}\cdot v,\end{aligned}\tag{5.7}$$

where we put $D_a=\sum_{P\in X}v_P(a_P)\cdot P\in\operatorname{Div}(X)$. In particular, when $a=\alpha$ is a constant idele, than the restriction of $R_X(\alpha)$ to the subspace $\mathscr{B}_X^D$ is $\alpha^{-2\deg D}\boldsymbol{I}$. Similarly, the representation $dR_X$ is given by

$$dR_X(x)(e_D\odot v)=-x(D)e_D\odot v+e_D\odot\boldsymbol{x}\cdot v,\quad x\in\mathbb{A}_X.$$

Let $\mathbb{J}_X^0=\{a\in\mathbb{J}_X:\deg D_a=0\}$ be the subgroup of degree 0 ideles, and let $G_X^0=\mathbb{J}_X^0\times k^*$ be the corresponding subgroup of $G_X$. In particular, the group of multiplicative functions $\mathcal{M}(X,D_{\mathrm{ns}})$, defined in Example 3.2, is a subgroup of $\mathbb{J}_X^0$. Restriction of the representation $R_X$ to $G_X^0$ preserves the subspace $\mathscr{B}_X$ of $\tilde{\mathscr{B}}_X$, and for $a\in\mathbb{J}_X^0$ the restriction of $R_X((a,1))$ to $\mathscr{B}_X$ is given by the same formula (5.7). In particular, constant ideles $a=\alpha$ act by identity in $\mathscr{B}_X$. From now on we will consider only this representation of $G_X^0$ in $\mathscr{B}_X$, and will continue to denote it by $R_X$.

Denote by $R_X^\vee$ the contragradient representation of $G_X^0$ in $\mathscr{B}_X^\vee$. We get from (5.7),

$$\begin{aligned}(q^D\odot u)\cdot\boldsymbol{a}&=((q^D\odot u)\cdot R_X^\vee((a,1))\\ &=(-1)^{\sum_{P\in X}v_P(D_a)v_P(D)}\prod_{P\in X}\alpha_P^{v_P(D_a)-2v_P(D)}q^{D-D_a}\odot u\cdot\exp\boldsymbol{\varphi},\end{aligned}\tag{5.8}$$

where we have used that $\sum_{P\in X}v_P(D)^2\equiv 0\bmod 2$ when $\deg D=0$.

**Theorem 5.4.** *There exists a unique linear functional $\langle\cdot\rangle:\mathscr{B}_X\to k$ — the expectation value functional — satisfying the following properties:*

- **EV1** $\langle\boldsymbol{1}_X\rangle=1$.
- **EV2** $\langle\boldsymbol{x}\cdot v\rangle=0$ *for all* $x\in\mathcal{A}(X,D_{\mathrm{ns}})$ *and* $v\in\mathscr{B}_X$.
- **EV3** $\langle\boldsymbol{m}\cdot v\rangle=\langle v\rangle$ *for all* $m\in\mathcal{M}(X,D_{\mathrm{ns}})$ *and* $v\in\mathscr{B}_X$.

*It has the form*

$$\langle v\rangle=(\boldsymbol{\Omega}_X,v),$$

*where*

$$\boldsymbol{\Omega}_X=\sum_{D\in\operatorname{Div}_0(X)}c(D)\,q^D\odot\exp\left\{\sum_{n=1}^\infty\sum_{P\in X}\eta_P^{(n)}(D)u_P^{(n)}\right\}\Omega_X\in\mathscr{B}_X^\vee.$$

*Here*

$$c(D)=\prod_{P,Q\in X}c(P,Q)^{v_P(D)v_Q(D)},\quad D\in\operatorname{Div}_0(X),$$

*where* $c(P,Q) = c_{P,Q} \in k^*$ *are given in Proposition* 3.2, and $\Omega_X$ — in Theorem 5.1.

*Proof.* It follows from the proof of Theorem 5.2 that conditions **EV1**–**EV2** ensure that $\boldsymbol{\Omega}_X$ has the form given above with some coefficients $c(D) \in k^*$. Since the constants act by identity in $\mathscr{B}_X$, it is sufficient to verify condition **EV3** for basic multiplicative functions $m = f_{P,Q}$. As in the proof of Theorem 5.2, we put

$$\Omega_D = \exp\left\{\sum_{n=1}^{\infty}\sum_{R\in X}\eta_R^{(n)}(D)u_R^{(n)}\right\}\Omega_X,$$

so that

$$\boldsymbol{\Omega}_X = \sum_{D\in\mathrm{Div}_0(X)} c(D)\, q^D \odot \Omega_D.$$

Using Lemma 3.4 and (5.8), we obtain the following formula for the action of $\boldsymbol{f_{P,Q}} = R_X^\vee(f_{P,Q})$ on $q^D \odot \Omega_D$,

$$(q^D \odot \Omega_D)\cdot \boldsymbol{f_{P,Q}} = h(P,Q;D)\, q^{D+Q-P} \odot \Omega_{D+Q-P},$$

where

$$h(P,Q;D) = (-1)^{v_P(D)+v_Q(D)}\prod_{R\in X}\left(\frac{c(P,R)}{c(Q,R)}\right)^{-2v_R(D)}\frac{c(P,P)c(Q,Q)}{c(P,Q)c(Q,P)}.$$

Now the equations

$$\boldsymbol{\Omega}_X \cdot \boldsymbol{f_{P,Q}} = \boldsymbol{\Omega}_X \quad \text{for all} \quad P,Q\in X,\ P\neq Q$$

are equivalent to the equations

$$c(D+Q-P) = h(P,Q;D)c(D) \quad \text{for all} \quad D\in\mathrm{Div}_0(X). \tag{5.9}$$

It is easy to see, using the property $c(P,Q) = -c(Q,P)$ for $P \neq Q$, that unique solution of recurrence relations (5.9) satisfying $c(0) = 1$ is given by

$$c(D) = \prod_{i,j=1}^{m} c(R_i,R_j)^{n_i n_j}, \quad \text{where} \quad D = \sum_{i=1}^{m} n_i R_i \in \mathrm{Div}_0(X). \qquad \square$$

*Remark* 5.2. All results of this section trivially hold for the case when $X$ has genus 0. Using Remarks 3.6, 3.8 and 3.10, one gets explicit elementary formulas for the expectation value functional $\langle\,\cdot\,\rangle$ for the quantum field theories of additive, charged, and multiplicative bosons on $\mathbb{P}^1_k$.

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